Asymptotics for the noncommutative Painlevé II equation

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Introduction

Full list of Painlevé equations

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = 6w^2 + z$$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = 2w^3 + zw + \alpha$$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = \frac{1}{w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{\alpha w^2 + \beta}{z} + \gamma w^3 + \frac{\delta}{w}$$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = \frac{1}{2w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 + \frac{3}{2}w^3 + 4zw^2 + 2(z^2 - \alpha)w + \frac{\beta}{w}$$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = \left(\frac{1}{2w} + \frac{1}{w-1}\right)\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{(w-1)^2}{z^2}\left(\alpha w + \frac{\beta}{w}\right) + \frac{\gamma w}{z} + \frac{\delta w(w+1)}{w-1}$$

Painlevé VI…

A short history of Painlevé equations

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$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = \mathcal{R}(z, w, \frac{\mathrm{d}w}{\mathrm{d}z}),$$

  which possess the Painlevé property.
- The solutions of Painlevé equations are called the Painlevé transcendents.

The Painelvé II equation

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$$q''(s) = 2q^3(s) + sq(s) - \nu.$$

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- $\nu = 0$: homogeneous PII equation.
- $\nu \neq 0$: inhomogeneous PII equation.
- All of its solutions are meromorphic in s whose poles are simple with residues ± 1.

Smooth solutions

- For any k, there exists a unique solution to the homogeneous PII equation which behaves like

$$k\mathrm{Ai}(s), \qquad s \to +\infty.$$

[Hastings-McLeod, '80]

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- By choosing different k, we obtain several classes of well-known solutions of the PII equation, which are denoted by $q(s;k)$ in what follows.

Hastings-McLeod solutions

$k = \pm 1$: Hastings-McLeod solutions

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- Asymptotic behaviors:

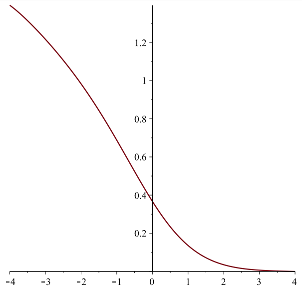
$$q(s;\pm 1) = \begin{cases} \pm\sqrt{-\frac{s}{2}} + O(|s|^{-5/2}), & s \to -\infty, \\ \pm \mathrm{Ai}(s) + O\left(\frac{e^{-(4/3)s^{3/2}}}{s^{1/4}}\right), & s \to +\infty. \end{cases}$$

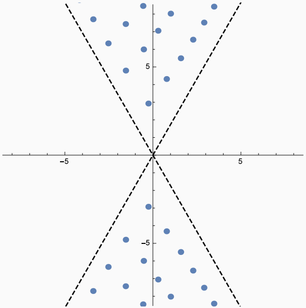
[Deift-Zhou, '95]

Hastings-McLeod solutions

- $q(s;\pm 1)$ are real and pole-free on the real axis.

[Hastings-McLeod, '80]





Ablowitz-Segur solutions

$-1 < k < 1$: Ablowitz-Segur solutions

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- Asymptotic behaviors:

$$q(s;k) = \begin{cases} \frac{\sqrt{-2\chi}}{(-s)^{1/4}} \cos\left(\frac{2}{3}(-s)^{3/2} + \chi \log(8(-s)^{3/2}) + \phi\right) \\ \qquad\qquad\qquad + O\left(\frac{\ln|s|}{|s|^{5/4}}\right), & s \to -\infty, \\ k\mathrm{Ai}(s) + O\left(\frac{e^{-(4/3)s^{3/2}}}{s^{1/4}}\right), & s \to +\infty, \end{cases}$$

where

$$\chi := \frac{1}{2\pi}\log(1-k^2), \quad \phi := -\frac{\pi}{4} - \arg\Gamma(i\chi) - \arg(-ki).$$

[Ablowitz-Segur, '76; Segur-Ablowitz, '81]
[Hastings-McLeod, '80; Clarkson-McLeod, '88; Deift-Zhou, '95]

Extensions of Ablowitz-Segur solutions

$k \in \mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty))$: (complex) Ablowitz-Segur solutions

- $q(s; k)$ is pole-free on the real line with the asymptotics

$$q(s;k) = \begin{cases} \frac{\sqrt{-2\chi}}{(-s)^{1/4}} \sin\left(\frac{2}{3}(-s)^{3/2} + \chi \ln(8(-s)^{3/2}) + \tilde{\phi}\right) + \mathcal{O}\left(\frac{1}{|s|^{2-3|\operatorname{Im}\chi|}}\right), & s \to -\infty, \\ k\mathrm{Ai}(s) + \mathcal{O}\left(\frac{e^{-(4/3)s^{3/2}}}{s^{1/4}}\right), & s \to +\infty. \end{cases}$$

Here, $\chi = \frac{1}{2\pi} \log(1 - k^2)$ with $|\operatorname{Im}\chi| < \frac{1}{2}$ and

$$\tilde{\phi} := -\frac{\pi}{4} - \frac{\mathrm{i}}{2} \ln \frac{\Gamma(-\mathrm{i}\chi)}{\Gamma(\mathrm{i}\chi)}.$$

[Bogatskiy-Claeys-Its, '16]

Applications: The Tracy-Widom distribution

Definition:

$$F_2(s;\gamma) = \det(I - \gamma\mathcal{K}_{\mathrm{Ai}}),$$

where $\mathcal{K}_{\mathrm{Ai}}$ is the integral operator acting on $L^2(s,\infty)$ with the Airy kernel

$$K_{\mathrm{Ai}}(x,y) = \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x-y},$$

i.e.,

$$F_2(s;\gamma) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_s^{\infty} \cdots \int_s^{\infty} \det(\gamma K_{\mathrm{Ai}}(\xi_i,\xi_j))_{i,j=1}^n \,\mathrm{d}\xi_1 \cdots \mathrm{d}\xi_n.$$

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- $F_2(s;\gamma)$ gives us the celebrated Tracy-Widom distribution ($\gamma = 1$) and its deformed version ($0 < \gamma < 1$).

Applications: The Tracy-Widom distribution

Integral representations:

$$F_2(s;\gamma) = \exp\left(\int_s^\infty -(x-s)q^2(x;\gamma)\,\mathrm{d}x\right) = \exp\left(-\int_s^\infty H(x;\gamma)\,\mathrm{d}x\right),$$

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where q satisfies Painlevé II equation

$$q''(x) = xq(x) + 2q^3(x),$$

subject to the following boundary conditions at $+\infty$:

$$q(x;\gamma) \sim \begin{cases} \mathrm{Ai}(x), & \gamma = 1 \text{ (Hastings-McLeod solution)}, \\ \sqrt{\gamma}\mathrm{Ai}(x), & 0 < \gamma < 1 \text{ (Ablowitz-Segur solution)}, \end{cases}$$

and H is the associated (scaled) Hamiltonian.

[Tracy-Widom, '94; Bohigas-Carvalho-Pato, '09]

Noncommutative Painlevé II equation

Noncommutative Painlevé II equation:

$$\mathbf{D}^2\beta_1 = 4\mathbf{s}\beta_1 + 4\beta_1\mathbf{s} + 8\beta_1^3, \quad \mathbf{s} := \mathrm{diag}(s_1, \dots, s_n), \quad \mathbf{D} := \sum_{j=1}^{n} \frac{\partial}{\partial s_j}, \quad n \in \mathbb{N},$$

where $\beta_1 = \beta_1(\vec{s})$ is an $n \times n$ matrix-valued function of $\vec{s} := (s_1, \dots, s_n)$.

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- If $n = 1$, one has $\beta_1(s_1) = \sqrt{2}q(2\sqrt{2}s_1)$.
- Introduced in the context of infinite Toda system.

[Retakh-Rubtsov, '10]

Noncommutative Painlevé II equation

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where $\beta_1 = \beta_1(\vec{s})$ is an $n \times n$ matrix-valued function of $\vec{s} := (s_1, \dots, s_n)$.

- Related to Tracy-Widom distribution function of the general β-ensembles with the even values of β.

  [Its-Prokhorov, '20; Rumanov, '15&'16]

- Related to the systems of Calogero type.

  [Bertola-Cafasso-Rubtsov, '18]

A family of special solutions for the noncommutative Painlevé II equation

Let $C = \left(c_{jk}\right)_{j,k=1}^{n}$ be an arbitrary $n \times n$ constant matrix, and set

$$S := \frac{1}{n}\sum_{i=1}^{n} s_j, \qquad \delta_i := s_i - S, \quad i = 1, \ldots, n.$$

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Theorem [Bertola-Cafasso, '12]

There exists a unique solution $\beta_1(\vec{s}) = \beta_1(\vec{s}; C)$ of the noncommutative Painlevé II equation such that

$$(\beta_1)_{kl} = -c_{kl}\mathrm{Ai}(s_k + s_l) + \mathcal{O}\left(\sqrt{S}e^{-\frac{4}{3}(2S-2\epsilon S)^{\frac{3}{2}}}\right), \qquad S \to +\infty,$$

with $|\delta_j| \leq \epsilon S$, where $\epsilon \in [0, 1)$ is an arbitrary real number and $(\beta_1)_{kl}$ stands for the (k, l)-th entry of β_1. If the singular values of C lie in $[0, 1]$, then the associated solution is pole free for $\vec{s} \in \mathbb{R}^n$.

Today's topic

- Asymptotics of $\beta_1(\vec{s}; C)$ as $S \to -\infty$ for a class of structured matrices C.

Main results

The structures of C

Assumption

We assume that $C = \Lambda P$, where $\Lambda = \mathrm{diag}(\mu_1, \ldots, \mu_n)$ with $\mu_i \in \mathbb{C}$, $|\mu_i| \leq 1, i = 1, \ldots, n$, and P is a permutation matrix such that C^2 is a diagonal matrix.

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Example

If $n = 3$, C takes one of the following forms:

$$\begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & \mu_1 \\ 0 & \mu_2 & 0 \\ \mu_3 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \mu_1 & 0 \\ \mu_2 & 0 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix}, \quad \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & 0 & \mu_2 \\ 0 & \mu_3 & 0 \end{pmatrix}.$$

Large negative S asymptotics of β_1

Theorem [Liu-Yao-Z., arXiv:2507.09472]

Under our assumption on C, we have

$$(\beta_1)_{kl} = \begin{cases} \sqrt{\frac{-s_k-s_l}{2}}c_{kl} + \mathcal{O}(S^{-1}), & c_{kl}c_{lk} = 1, \\ \frac{(-s_k-s_l)^{-\frac{1}{4}}}{\sqrt{\pi}} \cos\left(\mathrm{i}\left(\widehat{\theta}_k\left(\sqrt{\frac{s_k+s_l}{S}}\right) + \widehat{\theta}_l\left(\sqrt{\frac{s_k+s_l}{S}}\right)\right) - \frac{\pi}{4}\right) c_{kl} + \mathcal{O}(S^{-1}), & c_{kl}c_{lk} = 0, \\ (-s_k - s_l)^{-\frac{1}{4}}\sqrt{\frac{-\ln(1-c_{kl}c_{lk})}{\pi c_{kl}c_{lk}}} \cos\psi(s_k, s_l)c_{kl} + \mathcal{O}(S^{-1}), & c_{kl}c_{lk} \neq 0, \end{cases}$$

if $S \to -\infty$ and $\delta_i = \epsilon_i S$ with $\epsilon_i \in (-1, 1)$ being fixed,

Large negative S asymptotics of β_1

Theorem [Liu-Yao-Z., arXiv:2507.09472]

where the function $\psi(s_k, s_l)$ is related to the parameters c_{kl} and c_{lk} through the connection formula

$$\psi(s_k, s_l) := \mathrm{i}\left(\widehat{\theta}_k\left(\sqrt{\frac{s_k + s_l}{S}}\right) + \widehat{\theta}_l\left(\sqrt{\frac{s_k + s_l}{S}}\right)\right) + \frac{3}{4\pi}\ln(1 - c_{kl}c_{lk})\ln\left(-4(s_k + s_l)\right)$$
$$+ \frac{\mathrm{i}}{2}\ln\frac{\Gamma\left(-\frac{\ln(1-c_{kl}c_{lk})}{2\pi\mathrm{i}}\right)}{\Gamma\left(\frac{\ln(1-c_{kl}c_{lk})}{2\pi\mathrm{i}}\right)} + \frac{\pi}{4}$$

with

$$\widehat{\theta}_k(z) := \mathrm{i}(-S)^{\frac{3}{2}}\left(\frac{z^3}{6} - \frac{s_k}{S}z\right).$$

Remarks

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- If $n = 1$, we recover previous asymptotic formulas.
- If $n > 1$, special features of β_1.
  - $(\beta_1)_{kl}$ corresponds to either an extension of the Hastings-McLeod solution or an extension of the Ablowitz–Segur solution for the Painlevé II equation, depending on the value of the product $c_{kl}c_{lk}$.
  - Asymptotic behavior of$(\beta_1)_{kl}$ as $S \to -\infty$ cannot be deduced solely from its behavior as $S \to +\infty$ in the Ablowitz–Segur case.

Open problem

- Large negative S asymptotics of β_1 for general C.

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- Large negative S asymptotics of β_1 for general C.
- If C is a 2×2 Hermitian matrix with eigenvalues in $(-1, 1)$, Painlevé V asymptotics in a different asymptotic regime.

[Du-Xu-Zhao, '25]

About the proofs

A Fredholm determinant representation of β_1

Let $\mathcal{A}\boldsymbol{i}_{\vec{s}}$ be a matrix version of the Airy-convolution operator acting on $L^2(\mathbb{R}_+, \mathbb{C}^n)$ defined by

$$\left(\mathcal{A}\boldsymbol{i}_{\vec{s}}\vec{f}\right)(x) := \int_{\mathbb{R}_+} \mathbf{Ai}(x+y;\vec{s})\vec{f}(y)dy$$

with $\vec{f} := (f_1, \cdots, f_n)^T$ and

$$\mathbf{Ai}(x;\vec{s}) := \int_{\gamma_+} \mathrm{e}^{\theta(\mu)} C \mathrm{e}^{\theta(\mu)} \mathrm{e}^{\mathrm{i}x\mu} \frac{d\mu}{2\pi} = \left(c_{jk} \operatorname{Ai}(x + s_j + s_k)\right)_{j,k=1}^n,$$

where

$$\theta(\mu) := \mathrm{i}\, \mathrm{diag}\left(\frac{\mu^3}{6} + s_1\mu, \frac{\mu^3}{6} + s_2\mu, \ldots, \frac{\mu^3}{6} + s_n\mu\right).$$

A Fredholm determinant representation of β_1

A noncommutative (matrix) version of the Tracy-Widom distribution:

$$\det\left(I - \mathcal{A}\boldsymbol{i}^2_{\vec{s}}\right) = \exp\left(-4\int_s^\infty (t-s)\,\mathrm{Tr}\left(\beta_1(t+\vec{\delta})^2\right) dt\right).$$

[Bertola-Cafasso, '12]

A Riemann-Hilbert (RH) characterization of β_1

(1) $\Xi(\lambda) := \Xi(\lambda; \vec{s}, C)$ is defined and analytic in $\mathbb{C} \setminus (\gamma_+ \cup \gamma_-)$.

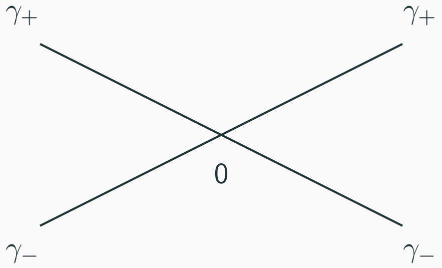

Figure 1: The jump contours γ_+ and γ_- in the RH problem for Ξ.

(b) For $\lambda \in \gamma_+ \cup \gamma_-$, we have

$$\Xi_+(\lambda) = \Xi_-(\lambda) \begin{pmatrix} I_n & e^{\theta(\lambda)} C e^{\theta(\lambda)} \chi_{\gamma_+} \\ e^{-\theta(\lambda)} C e^{-\theta(\lambda)} \chi_{\gamma_-} & I_n \end{pmatrix}.$$

(c) As $\lambda \to \infty$ with $\lambda \in \mathbb{C} \setminus (\gamma_+ \cup \gamma_-)$, we have

$$\Xi(\lambda) = I_{2n} + \frac{\Xi_1}{\lambda} + \mathcal{O}(\lambda^{-2}),$$

where Ξ_1 is independent of λ.

A Riemann-Hilbert (RH) characterization of β_1

The connection between β_1 and Ξ:

$$\beta_1(\vec{s}) = -\mathrm{i} \lim_{\lambda \to \infty} \lambda \, [\Xi]_{12} \, (\lambda; \vec{s}).$$

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Asymptotic analysis of Ξ for large S:

- $S > 0$: straightforward.
- $S < 0$: significant obstacles.

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Asymptotic analysis of Ξ for large S:

- $S > 0$: straightforward.
- $S < 0$: significant obstacles.

A key observation: the structures of C allow us to decompose the original RH problem into two RH problems by introducing proper index sets associated with the permutation matrix P.

Rescaling and decomposition of Ξ

- Rescaling:

$$\Psi(z) = \Xi(\sqrt{-S}z)e^{\widehat{\theta}(z)\otimes\sigma_3},$$

where

$$\widehat{\theta}(z) := \mathrm{i}(-S)^{\frac{3}{2}}\left[\frac{1}{6}z^3 I_n - \frac{z}{S}\mathbf{s}\right].$$

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- Two index sets: recall that $C = \Lambda P$, where $\Lambda = \mathrm{diag}(\mu_1, \ldots, \mu_n)$ and P is a permutation matrix such that C^2 is a diagonal matrix. Thus,

$$P = \sum_{i=1}^{n} E_{i\sigma(i)}, \qquad \sigma^2 = \mathrm{id}.$$

The decomposition of Ψ

- Two index sets: recall the permutation σ associated with P, we set

$$\mathcal{I} := \{i : c_{i\sigma(i)}c_{\sigma(i)i} = 1\}, \qquad \mathcal{J} = \{1, \ldots, n\} \setminus \mathcal{I}.$$

The decomposition of Ψ

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- The decomposition:

$$\Psi(z) = \Psi_{\mathcal{I}}(z) \begin{pmatrix} \sum_{i\in\mathcal{I}} E_{ii} & 0 \\ 0 & \sum_{i\in\mathcal{I}} E_{ii} \end{pmatrix} + \Psi_{\mathcal{J}}(z) \begin{pmatrix} \sum_{j\in\mathcal{J}} E_{jj} & 0 \\ 0 & \sum_{j\in\mathcal{J}} E_{jj} \end{pmatrix},$$

where

$$\Psi_{\mathcal{I}}(z) := \Psi(z) \begin{pmatrix} \sum_{i\in\mathcal{I}} E_{ii} & 0 \\ 0 & \sum_{i\in\mathcal{I}} E_{ii} \end{pmatrix} + \begin{pmatrix} \sum_{j\in\mathcal{J}} E_{jj} & 0 \\ 0 & \sum_{j\in\mathcal{J}} E_{jj} \end{pmatrix},$$

$$\Psi_{\mathcal{J}}(z) := \Psi(z) \begin{pmatrix} \sum_{j\in\mathcal{J}} E_{jj} & 0 \\ 0 & \sum_{j\in\mathcal{J}} E_{jj} \end{pmatrix} + \begin{pmatrix} \sum_{i\in\mathcal{I}} E_{ii} & 0 \\ 0 & \sum_{i\in\mathcal{I}} E_{ii} \end{pmatrix}.$$

RH problem for $\Psi_{\mathcal{I}}$

(a) $\Psi_{\mathcal{I}}(z)$ is defined and analytic in $\mathbb{C} \setminus (\gamma_+ \cup \gamma_-)$.

(b) For $z \in \gamma_+ \cup \gamma_-$, we have

$$\Psi_{\mathcal{I},+}(z) = \Psi_{\mathcal{I},-}(z) \begin{pmatrix} I_n & \sum_{i\in\mathcal{I}} E_{ii} C \chi_{\gamma_+} \\ \sum_{i\in\mathcal{I}} E_{ii} C \chi_{\gamma_-} & I_n \end{pmatrix}.$$

(c) As $z \to \infty$ with $z \in \mathbb{C} \setminus (\gamma_+ \cup \gamma_-)$, we have

$$\Psi_{\mathcal{I}}(z) = \left(I_{2n} + \frac{\Psi_{\mathcal{I},1}}{\sqrt{-Sz}} + \mathcal{O}\left(z^{-2}\right) \right) e^{\sum_{i\in\mathcal{I}} E_{ii} \widehat{\theta}(z) \otimes \sigma_3},$$

where

$$\Psi_{\mathcal{I},1} = \Xi_1 \begin{pmatrix} \sum_{i\in\mathcal{I}} E_{ii} & 0 \\ 0 & \sum_{i\in\mathcal{I}} E_{ii} \end{pmatrix}.$$

RH problem for $\Psi_{\mathcal{J}}$

(a) $\Psi_{\mathcal{J}}(z)$ is defined and analytic for $z \in \mathbb{C} \setminus (\gamma_+ \cup \gamma_-)$.

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$$\Psi_{\mathcal{J}}(z) = \left(I_{2n} + \frac{\Psi_{\mathcal{J},1}}{\sqrt{-Sz}} + \mathcal{O}\left(z^{-2}\right) \right) e^{\sum_{j \in \mathcal{J}} E_{jj} \widehat{\theta}(z) \otimes \sigma_3},$$

where

$$\Psi_{\mathcal{J},1} = \Xi_1 \begin{pmatrix} \sum_{j \in \mathcal{J}} E_{jj} & 0 \\ 0 & \sum_{j \in \mathcal{J}} E_{jj} \end{pmatrix}.$$

Asymptotic analysis of the RH problems for $\Psi_{\mathcal{I}}$ and $\Psi_{\mathcal{J}}$

- Asymptotic analysis of the RH problems for $\Psi_{\mathcal{I}}$
  - Introduction of g-function
  - Contour deformation
  - Airy parametrix

Asymptotic analysis of the RH problems for $\Psi_{\mathcal{I}}$ and $\Psi_{\mathcal{J}}$

- Asymptotic analysis of the RH problems for $\Psi_{\mathcal{I}}$
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  - Introduction of $\widetilde{\theta}$-function
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  - Lenses opening
  - Parabolic cylinder parametrix

Thanks for your attention!

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Questions?