Three topological phases of the elliptic Ginibre ensembles with a point charge

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joint work with Sung-Soo Byun

[arXiv:2502.02948]

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Log-gases in Caeli Australi

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Outline

1. Elliptic Ginibre Ensembles with a Point Charge

2. Sketch of Proof

3. Ongoing Projects
   - Multi–Component Droplet
   - Riemann–Hilbert Problem

Elliptic Ginibre Ensembles with a Point Charge

Logarithmic Potential Theory

- **2D Log-gas Ensemble** For $\boldsymbol{z} = (z_j)_{j=1}^N \in \mathbb{C}^N$

$$\mathbf{H}_N(\boldsymbol{z}) = \sum_{1 \le j < k \le N} \log \frac{1}{|z_j - z_k|} + \frac{N}{2} \sum_{j=1}^{N} W(z_j),$$

where $W(z) - \log(1 + |z|^2) \to \infty$.

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- **(Weighted) Logarithmic Energy Functional**

$$I_W(\mu) := \iint_{\mathbb{C}^2} \log \frac{1}{|z-w|}\, \mathrm{d}\mu(z)\, \mathrm{d}\mu(w) + \int_{\mathbb{C}} W(z)\, \mathrm{d}\mu(z),$$

- **Equilibrium Measure**

$$\frac{1}{N}\sum \delta_{z_j} \to \mathrm{d}\mu_W = \underset{\|\mu\|=1}{\operatorname{argmin}}\ I_W(\mu)$$

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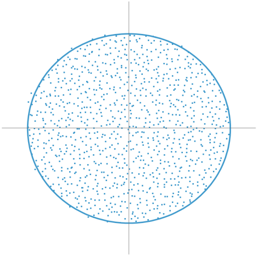
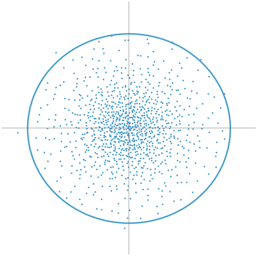
where $\Delta = \partial\bar{\partial}$, and $S_W \subset \mathbb{C}$ is called **droplet**.

Equilibrium Measure Problem



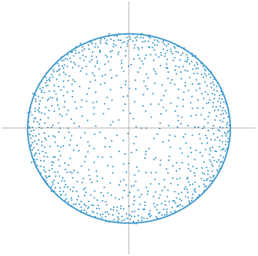
Ginibre UE

$W(z) = |z|^2$



Spherical UE

$W(z) = \rho^{-1} \log(1 + |z|^2)$



Truncated UE

$W(z) = \rho^{-1} \log(1 - |z|^2)$

Equilibrium Measure Problem

- **2D Equilibrium Measure Problems**
  - Ginibre Unitary Ensemble + Point Charges

    [Balogh–Bertola–Lee–McLaughlin '15], [Kieburg–Kuijlaars–Lahiry '25]
  - Spherical Unitary Ensemble + Point Charges

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  - Droplets with Discrete Rotational Symmetry

    [Balogh–Merzi '15], [Bleher–Kuijlaars '12], [Kuijlaars–del Rey '22]
  - Non-Hermitian Marchenko–Pastur Law (chiral Ginibre)

    [Akemann–Byun–Kang '21]

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- **Other Perspectives**
  - Riesz Gases

    [Agarwal–Dhar–Mulkarni–Kundu–Majumdar–Mukamel–Schehr '19]
  - Anisotropic Coulomb Interaction

    [Carrillo–Mateu–Mora–Rondi–Scardia–Verdera '19]
  - Random Matrices with Additive External Source

Elliptic Ginibre Ensemble

- **Elliptic Ginibre Matrix**

$$\mathbf{X}_\tau := \frac{\sqrt{1+\tau}}{2}(\mathbf{G}+\mathbf{G}^*) + \frac{\sqrt{1-\tau}}{2}(\mathbf{G}-\mathbf{G}^*)$$

where $\tau \in [0,1]$ and $\mathbf{G} =$ Ginibre.

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- **External Potential**

$$\begin{aligned} W^e(z) :=& \frac{1}{1-\tau^2}\left(|z|^2 - \tau \operatorname{Re} z^2\right) \\ =& \frac{x^2}{1+\tau} + \frac{y^2}{1-\tau} \end{aligned}$$

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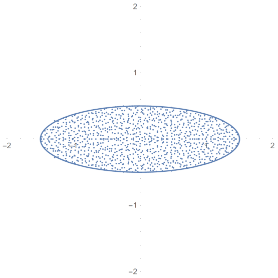
- **Limiting Spectrum**

$$\begin{cases} \tau = 0 & \text{Circular Law} \\ \tau \in (0,1) & \text{Elliptic Law} \\ \tau = 1 & \text{Semicircle Law} \end{cases}$$



The Elliptic Law
$(\tau = 0.5)$

- **Conditional Elliptic Ginibre Matrix**
  elliptic Ginibre matrix with non-Hermiticity parameter $\tau \in [0, 1]$

Elliptic Ginibre Ensemble with a Point Charge Insertion

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$$\mathrm{d}\mathbf{P}_N(\boldsymbol{z}) = \frac{1}{Z_N(Q)} \prod_{1 \le j < k \le N} |z_j - z_k|^2 \prod_{j=1}^{N} |z_j - p|^{2cN} e^{-NW^e(z_j)} \frac{\mathrm{d}^2 z_j}{\pi}$$

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- **External Potential**

$$\begin{aligned} Q(z) &= W^e(z) + 2c \log \frac{1}{|z-p|} \\ &= \frac{1}{1-\tau^2}(|z|^2 - \tau \operatorname{Re} z^2) + 2c \log \frac{1}{|z-p|} \\ &= \text{elliptic potential} \;+\; \text{point charge insertion} \end{aligned}$$

Main Goal

$$Q(z) = \frac{1}{1-\tau^2}(|z|^2 - \tau \operatorname{Re} z^2) + 2c \log \frac{1}{|z-p|}$$

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- **Equilibrium Measure**

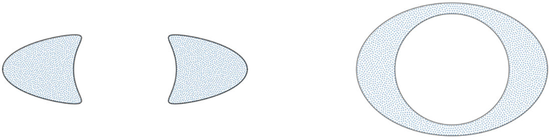
$$d\mu_Q = \Delta Q \cdot \mathbb{1}_{S_Q} \frac{d^2 z}{\pi} = \frac{1}{\pi(1-\tau^2)} \cdot \mathbb{1}_{S_Q}\, d^2 z$$

- **Parameters**
  - $p \geq 0$: position of the point charge insertion
  - $c \geq 0$: electrostatic charge
  - $\tau \in [0, 1]$: non-Hermiticity parameter

Previous Results

- $\tau = 0$ [Balogh–Bertola–Lee–McLaughlin '15]

- $p = 0$ [Byun '24]



Regimes of Parameters

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- **Regime I**

$$p \leq \min\left\{2\sqrt{\frac{2\tau(1+\tau)}{3+\tau^2}}, 2\sqrt{\frac{\tau(1-\tau-2c\tau)}{1-\tau}}\right\} \quad \text{and} \quad 0 \leq c \leq \frac{1-\tau}{2\tau},$$

or

$$2\sqrt{\frac{2\tau(1+\tau)}{3+\tau^2}} \leq p \leq (1+\tau)\sqrt{1+c} - \sqrt{c(1-\tau^2)} \quad \text{and} \quad 0 \leq c \leq \frac{(1-\tau)^3}{2\tau(3+\tau^2)}.$$

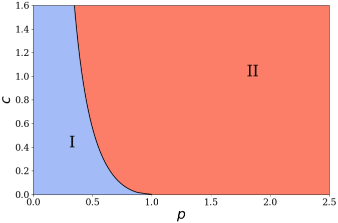
- **Regime II**

$$c \equiv c(a,\kappa) = \frac{\kappa}{a^2}\frac{(1-a^2)^2(1-\tau a^2)+a^2\kappa}{(1-a^2)^2(1-\tau^2+2\tau\kappa)-\kappa^2},$$

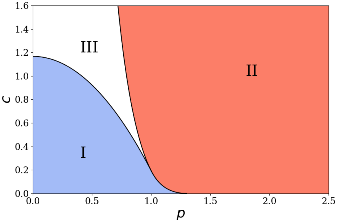
$$p \equiv p(a,\kappa) = \sqrt{\frac{1+\tau}{1-\tau}}\frac{(1-\tau)(1-a^2)(1+\tau a^2)-(1-\tau a^2)\kappa}{a\sqrt{(1-a^2)^2(1-\tau^2+2\tau\kappa)-\kappa^2}}.$$

Here, $a \in (0,1)$ and $\kappa \in [0, \kappa_{\mathrm{cri}})$ where κ_{cri} is the unique zero of $H(a,\cdot)$.

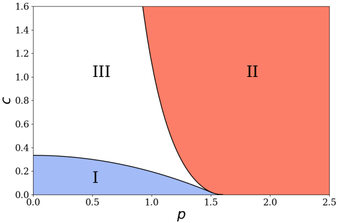
- **Regime III** Corresponds to (p, c, τ) lying outside **Regime I** and **II**.



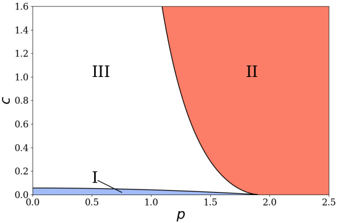
(a) $\tau = 0$



(b) $\tau = 0.3$



(c) $\tau = 0.6$



(d) $\tau = 0.9$

Topological Characterization of the Droplet

Theorem 1 (Byun–Y. '25)

The droplet S associated with Q is doubly connected, simply connected, or composed of two disjoint simply connected components.

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1. *S is doubly connected if and only if* $(p, c, \tau) \in$ **Regime I**
2. *S is simply connected if and only if* $(p, c, \tau) \in$ **Regime II**
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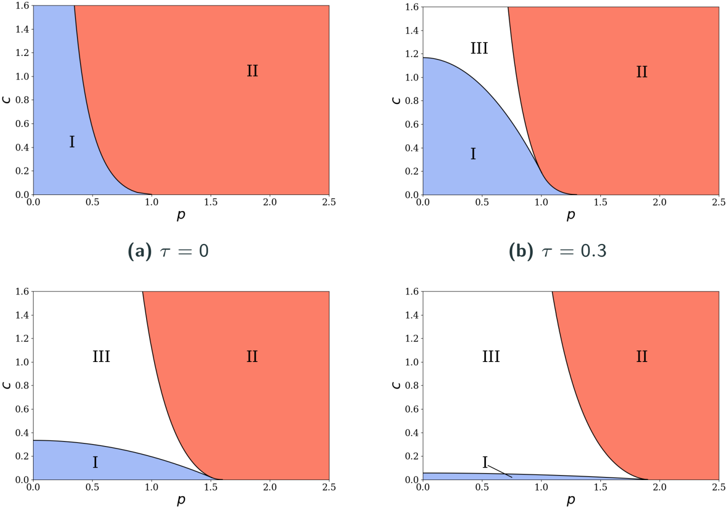
Phase Diagrams Revisited



(a) $\tau = 0$ **(b)** $\tau = 0.3$

(c) $\tau = 0.6$ **(d)** $\tau = 0.9$

Droplet and Energy: Doubly Connected Case

Theorem 2 (Byun–Y. '25)

Suppose $(p, c, \tau) \in$ **Regime I**. *Then* $S = E \cap D^c$ *where*

$$E = \Big\{ \Big(\frac{x}{1+\tau}\Big)^2 + \Big(\frac{y}{1-\tau}\Big)^2 \le 1 + c \Big\}, \quad D = \Big\{ (x-p)^2 + y^2 < c(1-\tau^2) \Big\}.$$

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The weighted logarithmic energy is given by $I_Q(\mu_Q) = \mathcal{I}_{\mathrm{d}}(p, c, \tau)$ *where*

$$\mathcal{I}_{\mathrm{d}}(p, c, \tau) := \frac{3}{4} + \frac{3c}{2} - \frac{c\,p^2}{1+\tau} + \frac{c^2}{2} \log\Big(c(1-\tau^2)\Big) - \frac{(1+c)^2}{2} \log(1+c).$$

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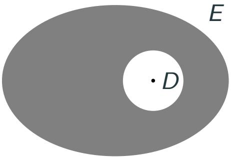
Remark.

– **Regime I** $\Leftrightarrow D \subset E$.



Droplet and Energy: Simply Connected Case

Theorem 3 (Byun–Y. '25)

Suppose $(p, c, \tau) \in$ **Regime II**. *Then* S *is closure of the interior of the Jordan curve* $f(\partial\mathbb{D})$ *where*

$$f(z) = R\left(z + \frac{\tau}{z} - \frac{\kappa}{z-a} - \frac{\kappa}{a(1-\tau)}\right).$$

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$$\begin{aligned}\mathcal{I}_{\rm s}(p,c,\tau) &:= \frac{3}{4} + \frac{3c}{2} - \frac{cp^2}{1+\tau} \\ &\quad + \frac{R^3\kappa p(2-3a^2-3\tau a^2+2\tau a^4)}{2(1-\tau^2)^2a^3}\Big(1-\tau-\frac{2-3a^2+3\tau a^2-2\tau a^4}{2-3a^2-3\tau a^2+2\tau a^4}\frac{\kappa}{1-a^2}\Big) \\ &\quad + 2c(1+c)\log a + c^2\log\Big(\frac{c(1-\tau^2)(1-a^2)}{R\kappa}\Big) - (1+c)^2\log R.\end{aligned}$$

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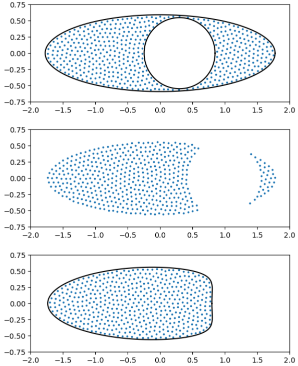
- **Moments of Characteristic Polynomials**
  [Webb–Wong '19], [Deaño–Simm '22], [Byun–Seo–Yang '25], [Deaño–McLaughlin–Molag–Simm '25], [Byun–Forrester–Lahiry '25]

- **Fekete Points** Minimizing configuration of Hamiltonian $\mathbf{H}_N$



$p = 0.3$ **doubly connected**

$p = 1.0$ **two simply connected components**

$p = 1.5$ **simply connected**

Fekete Points for $c = 0.4, \tau = 0.5$

The Number of Phases and Duality

Recall that

$$W(z) = \rho^{-1} \log(1 \pm |z|^2), \quad \text{where } \pm = \begin{cases} + & \text{SrUE}, \\ - & \text{TrUE}, \end{cases}$$

$$W(z) = \frac{1}{1-\tau^2}(|z|^2 - \tau \operatorname{Re} z^2), \quad \text{for elliptic GinUE.}$$

<table>
<tr><th></th><th>GinUE</th><th>SrUE</th><th>TrUE</th><th>elliptic GinUE</th></tr>
<tr><td></td><td colspan="4">+ point charge insertion</td></tr>
<tr><td># of parameters</td><td>2</td><td colspan="3">3</td></tr>
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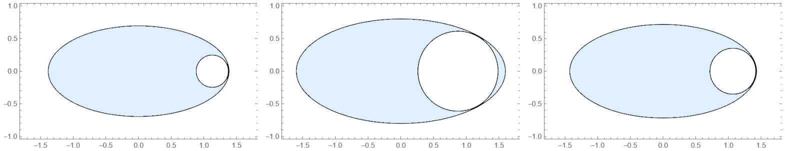
	GinUE	**SrUE**	**TrUE**	**elliptic GinUE**
	+ point charge insertion			
# of parameters	2	3		
# of topological phases	2			3

Remark. Dualities in Random Matrix Theory [Forrester '25]

- **GinUE $\leftrightarrow$ LUE**
- **SrUE, TrUE $\leftrightarrow$ JUE**



(a) Regime I ∩ II **(b) Regime I ∩ III** **(c) Triple Point**

- **Regime I ∩ II**: single tangent point.
- **Regime I ∩ III**: two tangent points.
- **Regime II ∩ III**: 2D *birth of a cut.*
- **Triple Point**: single tangent point + identical curvature.

$$c_{\mathrm{tri}} = \frac{(1-\tau)^3}{2\tau(3+\tau^2)}, \qquad p_{\mathrm{tri}} = 2\sqrt{\frac{2\tau(1+\tau)}{3+\tau^2}}.$$

Sketch of Proof

Outline of the Proof

- **Connectivity Bound**
  - Quadrature Domain (QD) theory
  - Lee–Makarov theorem [Lee–Makarov '16]

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- **Doubly Connected** $\Leftrightarrow$ **Regime I**
  - [$\Rightarrow$] conformal mapping method
  - [$\Leftarrow$] variational conditions (proven in [Byun '24])
- **Simply Connected** $\Leftrightarrow$ **Regime II**
  - [$\Rightarrow$] conformal mapping method
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- **Two Simply Connected Components $\Leftrightarrow$ Regime III**

Lee–Makarov Theorem and Conformal Mapping Method

Lee–Makarov, *Topology of quadrature domains*, J. Amer. Math. Soc. **29** (2016), 333–369.
Lee–Makarov, *Sharpness of connectivity bounds for quadrature domains*, arXiv:1411.3415.

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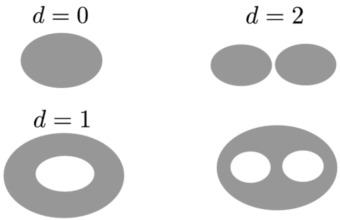

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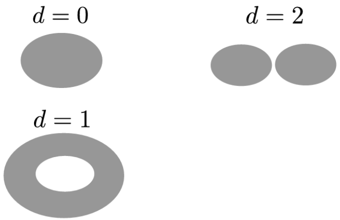

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Fact. [Aharonov–Shapiro '76]

Ω is a *simply connected* QD $\Leftrightarrow$ Ω has a rational Riemann map

Variational Conditions

For (p, c, τ) fixed, *Assume* that the droplet is simply connected...

- **Ansatz**

$$f : \bar{\mathbb{D}}^c \to \Omega = K^c, \quad \mathrm{d}\mu_K = \frac{1}{\pi(1-\tau^2)} \mathbb{1}_K \, \mathrm{d}^2 z$$

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- **Euler–Lagrange Conditions**

$$\mathcal{U}_K(z) := \int_{\mathbb{C}} \log \frac{1}{|z-w|} \, \mathrm{d}\mu_K(w) + \frac{1}{2} Q(z) \begin{cases} = \ell & z \in K, \\ > \ell & z \notin K. \end{cases}$$

Remark.

– Equality condition ⇔ QD condition

Ongoing Projects

Multi-Component Droplets

- **Single-Component vs Multi-Component Droplets**
  - Asmyptotics of Orthogonal Polynomials [Hedenmalm–Wennman '21]
  - Free Energy Expansion [Zabrodin–Wiegmann '06], [Byun '25]
  - Quadrature Domain [Aharonov–Shapiro '76]

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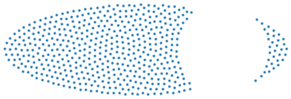

[Byun–Y. '25]

(Joint work with **S.–S. Byun** and **S.–Y. Lee**)

- **Ansatz** $f : \mathbb{U} \to S^c$ where

$$f(z) = \alpha\Big(\zeta(z - z_0) - \tau\zeta(z + z_0) - (1 - \tau)\zeta(z + \bar{z}_1)\Big) + \beta$$

where ζ is the Weierstrass elliptic function of periods $2, 2\omega$.

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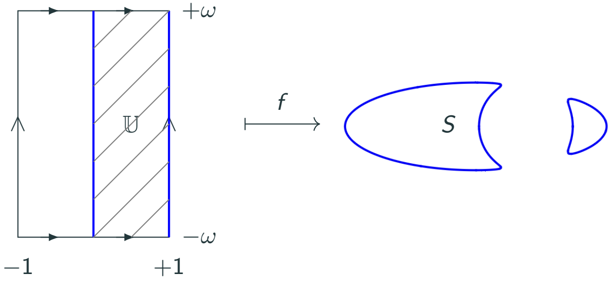

Two Critical Cases

- **Critical Cases**
  - Two Components $\rightarrow$ Doubly Connected when $|\omega| \rightarrow \infty$
  - Two Components $\rightarrow$ Simply Connected when $|\omega| \rightarrow 0$

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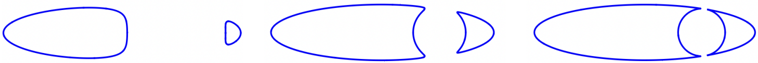

(a) $|\omega| = 0.8$ **(b)** $|\omega| = 1.5$ **(c)** $|\omega| = 2.5$

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  - Two Components $\rightarrow$ Doubly Connected when $|\omega| \rightarrow \infty$
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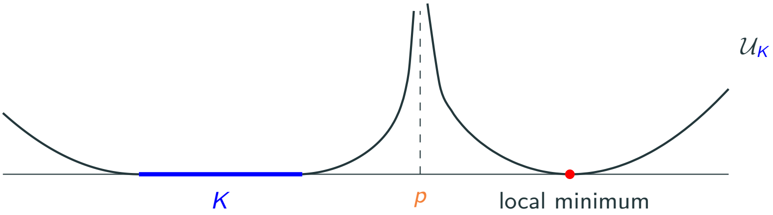

Riemann–Hilbert Problem

(Joint work with **S.–S. Byun**, **S.–Y. Lee**, and **M. Yang**)

$$Q(z) = \frac{1}{1-\tau^2}(|z|^2 - \tau \operatorname{Re} z^2) + 2c \log \frac{1}{|z-p|}$$

- **Planar Orthogonality**

$$\int_{\mathbb{C}} P_{n,N}(z)\overline{P_{m,N}(z)} e^{-NQ(z)} \, \mathrm{d}^2 z = h_n \delta_{n,m}$$

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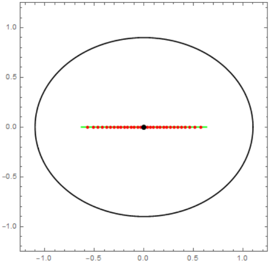
- **Contour Orthogonality**

$$\int_{\mathbb{R}} P_{n,N}(x)\omega_{m,N}(x)\,\mathrm{d}x = 0, \quad m = 0, \dots, N-1,$$

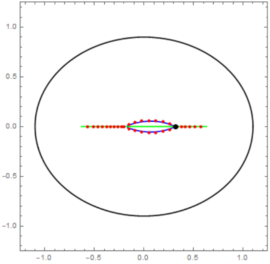
$$\omega_{m,N}(x) = (x-p)^{cN} D_{m+cN}\left(\sqrt{\frac{N}{\tau(1-\tau^2)}}(x-\tau p)\right) \exp\left(\frac{N(x-\tau p)^2}{4\tau(1-\tau^2)} - \frac{Nx^2}{2\tau}\right).$$

where $D_\nu(z)$ is the parabolic cylinder function (Weber–Hermite function).

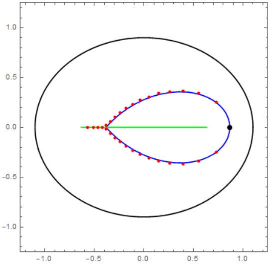
Remark. Planar Orthogonality of Hermite Polynomials [van Eijndhoven–Meyers '90]



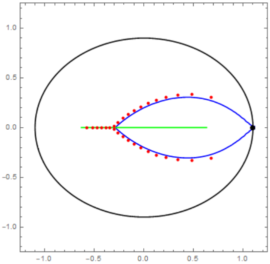
(a) $p = 0$



(b) $p = \sqrt{\tau} \asymp 0.32$



(c) $p = \sqrt{\tau} + (1 + \tau)/2 \asymp 0.87$



(d) $p = 1 + \tau$

Zeros of $P_{N,N}$ for $N = 30, cN = 1, \tau = 0.1$.

Riemann–Hilbert Problem

- **Type II Multiple Orthogonality**

$$\int_{\mathbb{R}} P_{n,N}(x) x^j w_i(x)\,\mathrm{d}x = 0, \quad j = 0, \ldots, n_i - 1, \quad i = 1, 2,$$

$$w_1(x) = \omega_{\lfloor \frac{n-1}{2} \rfloor, N}(x), w_2(x) = \omega_{\lfloor \frac{n+1}{2} \rfloor, N}(x), \quad n_1 = \lfloor \frac{n+1}{2} \rfloor, n_2 = \lfloor \frac{n}{2} \rfloor.$$

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- **Type I Multiple Orthogonality** for $cN =$ integer,

$$\int_{\mathbb{R}} \left(P_{n,N}(x)(x-a)^{cN} e^{-\frac{Nx^2}{2\tau}} + Q_{n,N}(x) e^{-\frac{N(x-\tau p)^2}{2\tau(1-\tau^2)}} \right) x^m\,\mathrm{d}x = 0,$$

$$\deg Q_{n,N} \leq cN - 1, \quad m = 0, \ldots, n + cN - 1.$$

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- **Future Directions**
  - Correlation Kernel at the Point of Small Insertion
    [Akemann et al. '97], [Kuijlaars–Vanlessen '03], [Lee–Yang '17]
  - Triple Criticality (Painlevé Hierarchy)
    [Krüger–Lee–Yang '25]
  - Free Energy Expansion (Oscillatory Behavior)
    [Ameur–Charlier–Cronvall '23], [Byun '25]

Summary

- Droplets for elliptic Ginibre ensembles with a point charge are:
  - Doubly Connected $\Leftrightarrow$ **Regime I**
  - Simply Connected $\Leftrightarrow$ **Regime II**
  - Two Components $\Leftrightarrow$ **Regime III**
- Quadrature Domain Theory and Variational Conditions
- Description of Two Component Droplet by Elliptic Functions
- Formulation of the corresponding Riemann–Hilbert Problem


Thank you for your attention!

Quadrature Domain

- **Quadrature Domain** Domain Ω is a *QD* if the **quadrature identity**

$$\int_{\Omega} f(\zeta)\frac{\mathrm{d}^2\zeta}{\pi} = \sum_{k=1}^{n} c_k f^{(m_k)}(a_k) = \frac{1}{2\pi i}\int_{\partial\Omega} f(\zeta) r_{\Omega}(\zeta)\,\mathrm{d}\zeta$$

holds for all integrable analytic f on Ω. Here

$$r_{\Omega}(\zeta) := \sum_{k=1}^{n} \frac{c_k m_k!}{(\zeta - a_k)^{m_k+1}}$$

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- **Order of a QD** $\Omega := \deg r_{\Omega}$
- **Mean Value Property of** $\mathbb{D} \subset$ **QD**

$$\int_{\mathbb{D}} f(\zeta)\frac{\mathrm{d}^2\zeta}{\pi} = f(0) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} f(\zeta)\frac{1}{\zeta}\, d\zeta.$$

- **Algebraic Hele–Shaw Potentials**

$$\frac{1}{t}W(\zeta) = |\zeta|^2 - H(\zeta), \quad h(\zeta) = \partial H(\zeta) : \text{rational in } \zeta.$$

Theorem 4 (Lee–Makarov)

Let $S \equiv S_W$ be the droplet respect to an algebraic Hele–Shaw potential W.

Topology of QD

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Let d be the degree of h and q_j be the number of QDs with connectivity $j \geq 1$. Then

$$\#(ovals) + q_{odd} + 4(q - q_1) \leq 2d + 2,$$

where $q = \sum_j q_j$, and $q_{odd} = \sum_{j=odd} q_j$.

In particular, above connectivity bound is sharp.

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$$(1-\tau^2)Q(\zeta) = |\zeta|^2 - \tau \operatorname{Re}\zeta^2 - 2c(1-\tau^2)\log|\zeta - p|,$$

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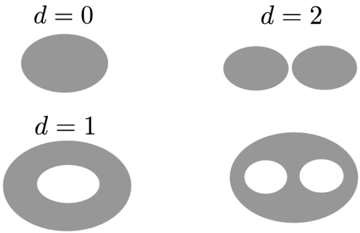

Connectivity Bound

Droplets and Quadrature Domains

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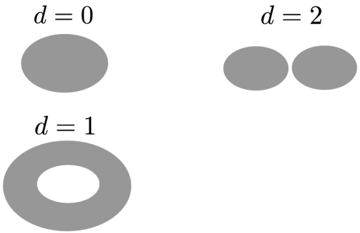

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  - **Doubly Connected Case**

$$r_{\Omega_1}(\zeta) = \tau\zeta, \quad r_{\Omega_2}(\zeta) = \frac{c(1-\tau^2)}{\zeta - p}, \quad \Omega_1 \cup \Omega_2 = S^c$$

  - **Simply Connected Case/Two Component Case**

$$r_\Omega(\zeta) = h(\zeta), \quad \Omega = S^c = \text{Simply/Doubly Connected}$$

If Ω is a quadrature domain then

$$\bar{\zeta} = r_{\Omega}(\zeta) + C_{\Omega^c}(\zeta), \quad \zeta \in \partial\Omega$$

where C_{Ω^c} is the Cauchy transform respect to $\mathbb{1}_{\Omega^c}\, \mathrm{d}^2\zeta/\pi$.

Conformal Mapping Method

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Fact. [Aharonov–Shapiro '76]

Ω is a simply connected QD $\Leftrightarrow$ Ω has a rational Riemann map

For a *simply connected* Ω,

$$\overline{f(1/\bar{z})} = r_\Omega(f(z)) + C_{\Omega^c}(f(z)), \quad f : \bar{\mathbb{D}}^c \to \Omega \ \text{ rational, conformal}$$

Corollary 5 (Byun–Y. '25)

Let $\mathbf{X} \equiv \mathbf{X}_\tau$ *be elliptic Ginibre matrices of size* $N \times N$*.*
Let $c > 0$, $\tau \in [0, 1)$, $z \in \mathbb{R}$ *be fixed.*

Application: Moments of the Characteristic Polynomials

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Let $c > 0$, $\tau \in [0,1)$, $z \in \mathbb{R}$ *be fixed. Then as* $N \to \infty$,

$$\log \mathbb{E}\Big[\Big| \det(\mathbf{X} - z) \Big|^{2cN} \Big] = \begin{cases} \mathcal{K} N^2 + \mathcal{E}_N, & \text{for the complex case} \\ 2\,\mathcal{K} N^2 + \mathcal{E}_N, & \text{for the symplectic case} \end{cases}$$

where

$$\mathcal{K} = -I_Q(\mu_Q) + \frac{3}{4} = \begin{cases} -\mathcal{I}_{\mathrm{d}}(z,c,\tau) + \frac{3}{4} & \text{if } \quad (p = z, c, \tau) \in \textbf{Regime I}, \\ -\mathcal{I}_{\mathrm{s}}(z,c,\tau) + \frac{3}{4} & \text{if } \quad (p = z, c, \tau) \in \textbf{Regime II}. \end{cases}$$

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Here, the error term is given by

$$\mathcal{E}_N = \begin{cases} o(N^{1/2+\epsilon}) & \text{for the complex case,} \\ o(N^2) & \text{for the symplectic case.} \end{cases}$$

Application: Moments of the Characteristic Polynomials

- **Partition Function and Characteristic Polynomial**

$$\mathbb{E}\Big[\Big|\det(\mathbf{X} - z)\Big|^{2cN}\Big] = Z_N(Q)/Z_N(W^e)$$

where

$$W^e(\zeta) = \frac{1}{1-\tau^2}(|\zeta|^2 - \tau \operatorname{Re}\zeta^2), \quad Q(\zeta) = W^e(\zeta) - 2c \log|\zeta - z|.$$

- **Free Energy Expansions**

Based on

$$\log Z_N^{\mathbb{C}}(W) = -I_W(\mu_W)N^2 + o(N^2),$$
$$\log Z_N^{\mathbb{H}}(W) = -2I_W(\mu_W)N^2 + o(N^2),$$

The leading term $\mathcal{K}$ is given by

$$\mathcal{K} = -I_Q(\mu_Q) + I_{W^e}(\mu_{W^e}) = -I_Q(\mu_Q) + \frac{3}{4}.$$

Free Energy Expansion and Topology

Remark.

- **Topological Dependence**
  The coefficient of $\log N$ term is expected to be

$$\begin{cases} \frac{1}{2} - \frac{1}{12}\chi(S_W) & \text{for the complex case} \\ \frac{1}{2} - \frac{1}{24}\chi(S_W) & \text{for the symplectic case} \end{cases}$$

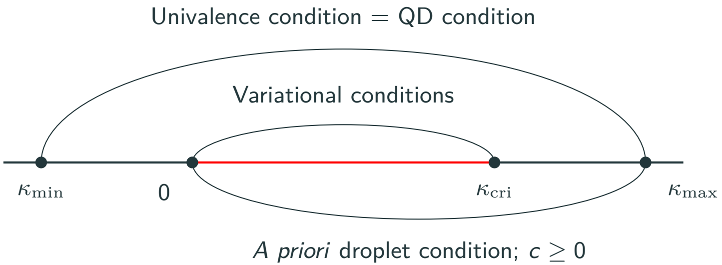
  [Jancovici–Manificat–Pisani '94], [Téllez–Forrester '99], [Byun–Kang–Seo '23]

- **Ginibre Case** For $\gamma = O(1)$ or cN and $|z| < 1$,

$$\mathbb{E}\Big[\big|\det(\mathbf{G} - z)\big|^{2\gamma}\Big] = N^{-\gamma N} e^{\gamma N|z|^2} \frac{G(1+\gamma+N)}{G(1+\gamma)G(1+N)}(1+o(1)).$$

  [Webb–Wong '19], [Deaño–Simm '22], [Byun–Seo–Yang '24]

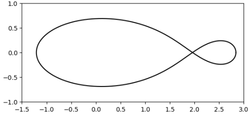
- **Other Models**
  [Fyodorov '16], [Fyodorov–Tarnowski '21], [Kivimae '24]
  [Deaño–McLaughlin–Molag–Simm '25]



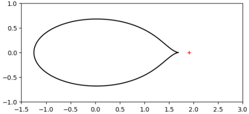
Remark.

- Condition $c \geq 0$ gives $\kappa \geq 0$.
- Condition $p \geq 0$ gives $a \in (0, 1)$.

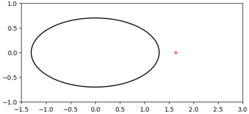
A Priori Droplet Condition



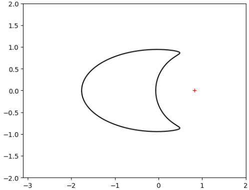
(a) $\kappa < \kappa_{\min}$



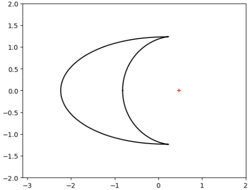
(b) $\kappa = \kappa_{\min}$



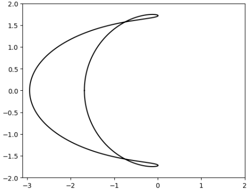
(c) $\kappa = 0$



(d) $\kappa = \kappa_{\mathrm{cri}}$



(e) $\kappa = \kappa_{\max}$



(f) $\kappa > \kappa_{\max}$