Planar Orthogonal polynomials and Their Applications

Meng Yang

Great Bay University
my@gbu.edu.cn

Log-gases in Caeli Australi, Aug.4-15, 2025

Joint works with Seung-Yeop Lee(USF), Torben Krüger(FAU), Sung-Soo Byun(SNU), Seong-Mi Seo(CNU).

2D Coulomb Gases

In the two-dimensional Coulomb gas model, we consider n particles as a system of point charges with the same sign located at points $\{z_j\}_{j=1}^n \in \mathbb{C}$, influenced by an external potential Q. We increase the number of point charges and the external potential such that in the scaling limit ($n \to \infty, N \to \infty$, while n/N is fixed), all the point charges are condensed to a compact set in $\mathbb{C}$, which we call the **droplet** S_Q. The probability distribution is given by

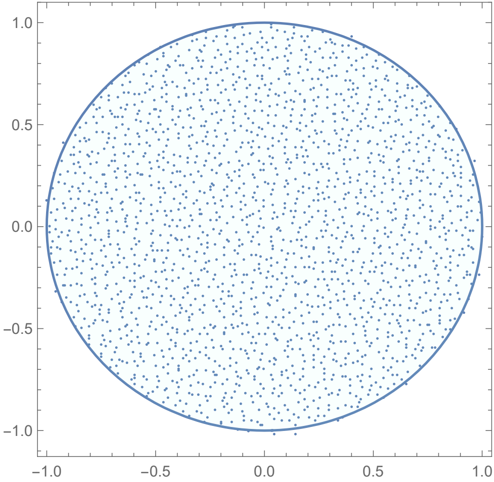
$$\mathrm{d}\mathbf{P}_n = \frac{1}{\mathcal{Z}_n} \prod_{i<j} |z_i - z_j|^2 \cdot \exp\left(- N \sum_{j=1}^n Q(z_j) \right) \cdot \prod_{j=1}^n \mathrm{d}A(z_j),$$

where $\mathrm{d}A$ denotes the standard Lebesgue measure on the plane.

Some Examples

Ginibre Ensemble

$$Q(z) = |z|^2.$$



Ginibre 1965, etc.

Some Examples

Elliptic Ginibre Ensemble

Girko 1985, Sommers-Crisanti-Sompolinsky-Stein 1988, etc.

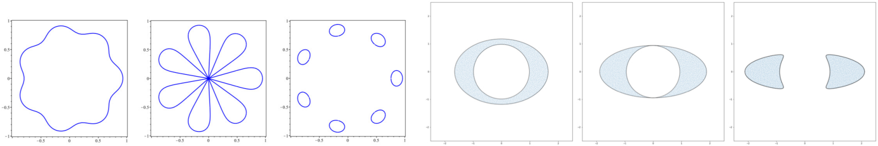
Lemniscate Ensemble

Balogh-Merzi 2013, Bertola-Elias Rebelo-Grava 2018.

Elliptic Ginibre Ensemble with One Insertion

Byun 2023, Byun-Yoo 2025

......



Byun-Forrester 2025

Planar Orthogonal Polynomials

A connection to orthogonal polynomials can be provided by *Heine's formula* i.e.,

$$p_n(z) = \mathbb{E} \prod_{j=1}^{n} (z - z_j).$$

Here $p_n(z)$ satisfies the orthogonality condition,

$$\int_{\mathbb{C}} p_n(z)\overline{p_m(z)} e^{-NQ(z)} \mathrm{d}A(z) = h_n \delta_{nm} \quad (n, m = 0, 1, 2, \cdots).$$

The First General Result

The external potential satisfies the general settings.
When $z \notin S_Q$, the following asymptotics holds.

$$p_n(z) \sim N^{\frac{1}{4}} \sqrt{\phi'_\tau(z)} [\phi_\tau(z)]^n \mathrm{e}^{N\mathcal{Q}_\tau(z)} \left(\mathcal{B}_{\tau,0}(z) + \frac{\mathcal{B}_{\tau,1}(z)}{N} + \dots \right).$$

Hedenmalm-Wennman 2021

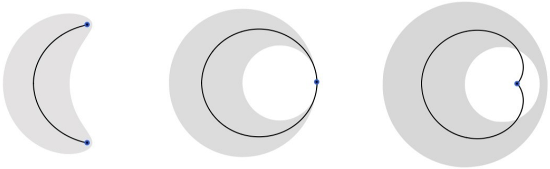
Ginibre Ensemble with (big) Insertions

$$Q(z) = |z|^2 - 2c \log |z - a|,$$

where $c > 0$ and $a \neq 0, \infty$.

Balogh-Bertola-Lee-McLaughlin 2015.

The droplet:

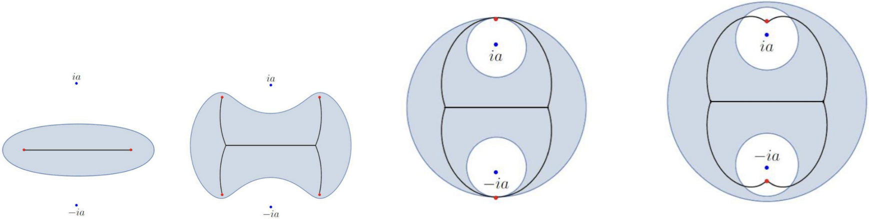


Ginibre Ensemble with (big) Insertions

$$Q(z) = |z|^2 - 2c \log|z - ia| - 2c \log|z + ia|.$$

Kieburg-Kuijlaars-Lahiry 2025.

The droplet:



Ginibre Ensemble with one (small) Insertion

We consider the external potential,

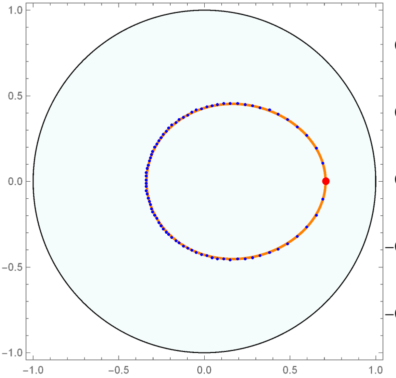
$$Q(z) = |z|^2 + \frac{2c_1}{N} \log \frac{1}{|z - a_1|},$$

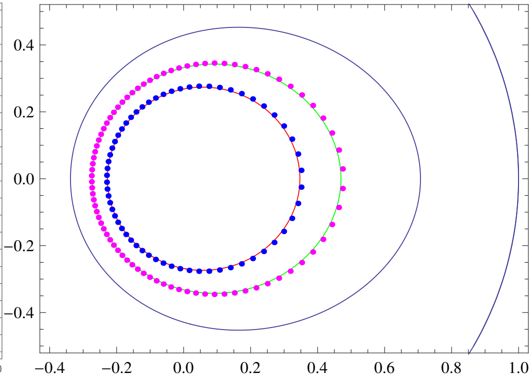
where $c_1 > -1$ and $a_1 \neq 0$.

$$\int_{\mathbb{C}} p_n(z)\overline{p_m(z)} \mathrm{e}^{-N|z|^2} |z - a_1|^{2c_1} \mathrm{d}A(z) = h_n \delta_{nm} \quad (n, m = 0, 1, 2, \cdots),$$

where the branch cut of $(z - a_1)^{c_1}$ is $[0, a_1]$.

When $\nu = 1$, the zeros of orthogonal polynomials for $c_1 = 1$ and $a_1 = \sqrt{2}/2$ (left). The zeros of orthogonal polynomials for $c_1 = e^{-\eta n}$, where $\eta = 0.4$ (blue) and $\eta = 0.2$ (magenta)(right).





Theorem (Lee-Yang 2017)

For $a_1 < 1$ and for any fixed nonzero $c_1 > -1$, we have

$$p_n(z) = \begin{cases} z^n \left(\frac{z}{z-a_1}\right)^{c_1} \left(1 + \mathcal{O}(\frac{1}{N^\infty})\right), & z \in \Omega_0, \\ -\frac{a_1^{n+1}(1-a_1^2)^{c_1-1}}{N^{1-c_1}\Gamma(c_1)} \frac{e^{Na_1(z-a_1)}}{z-a_1} \left(1 + \mathcal{O}(\frac{1}{N})\right), & z \in \Omega_1. \end{cases}$$

When $c_1 \in \mathbb{Z}$, orthogonal polynomials were studied. Akemann-Vernizzi 2003.

Ginibre Ensemble with multiple (small) Insertions

We consider the external potential,

$$Q(z) = |z|^2 + 2\sum_{j=1}^{\nu} \frac{c_j}{N} \log \frac{1}{|z - a_j|},$$

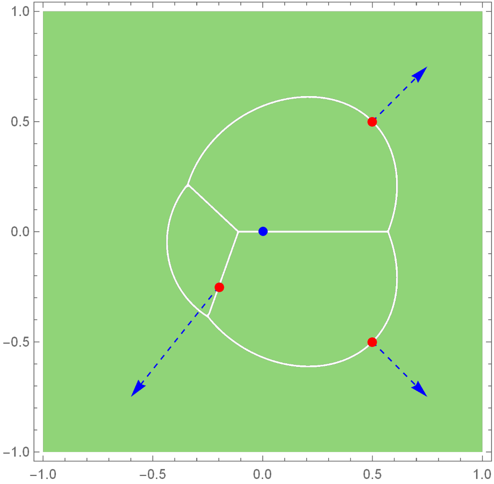
where $\{c_1, \dots, c_\nu\}$ are nonzero real numbers greater than -1 and $\{a_1, \dots, a_\nu\}$ are distinct points in $\mathbb{D} \setminus \{0\}$.

$$\int_{\mathbb{C}} p_n(z)\overline{p_m(z)}\mathrm{e}^{-N|z|^2}|W(z)|^2 \mathrm{d}A(z) = h_n \delta_{nm} \quad (n, m = 0, 1, 2, \cdots),$$

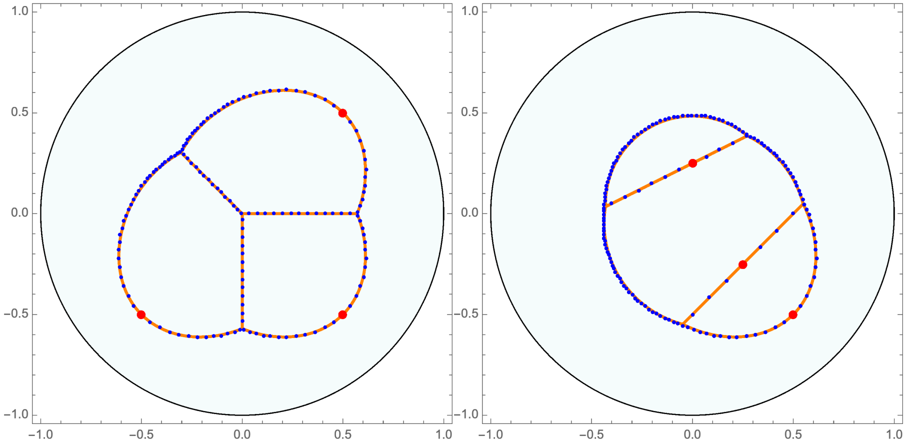
where $W(z) = \prod_{j=1}^{\nu} (z - a_j)^{c_j}$.

Branch cuts

The branch cuts of $W(z)$ are $\mathbf{B} = \bigcup_{j=1}^{\nu} \mathbf{B}_j, \mathbf{B}_j = \{a_j t, t \geq 1\}$.

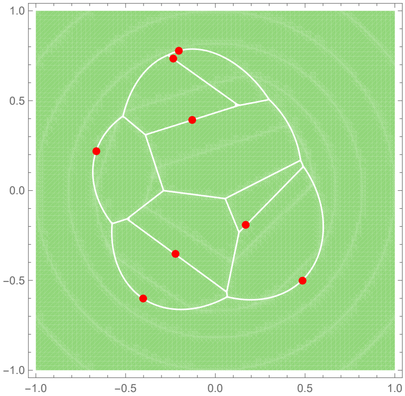


When $\nu = 3$, the zeros of orthogonal polynomials for $c_1 = c_2 = c_3 = 1$, $a_1 = 0.5 + 0.5\mathrm{i}, a_2 = -0.5 - 0.5\mathrm{i}, a_3 = 0.5 - 0.5\mathrm{i}$(left) and $a_1 = 0.25\mathrm{i}, a_2 = 0.25 - 0.25\mathrm{i}, a_3 = 0.5 - 0.5\mathrm{i}$(right). The limiting locus.

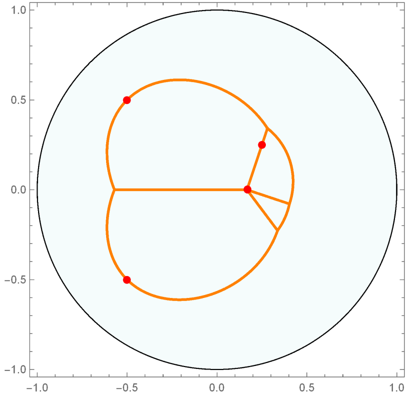


Multiple Szegő curves

Multiple Szegő curve: $\Gamma = \bigcup_{j=1}^{\nu} \partial\Omega_j$



non-generic case



Theorem (Lee-Yang 2023)

As $n \to \infty$ such that $n/N = 1$ the polynomial p_n satisfies

$$p_n(z) = \begin{cases} \dfrac{z^{n+\sum c_j}}{W(z)} \left(1 + \mathcal{O}\left(\dfrac{1}{N^\infty}\right)\right), & z \in \Omega_0, \\ -\dfrac{\exp\left[N(\overline{a}_j z + \ell_j)\right](z - a_j)^{c_j}}{W(z)} \dfrac{\textbf{chain}(j)}{z - a_j} \left(1 + \mathcal{O}\left(\dfrac{1}{N}\right)\right), & z \in \Omega_j, \end{cases}$$

where **chain**(j) *and* ℓ_j *are explicit constants with respect to* a_j.

Strategy of the Proof

Lemma (Lee-Yang 2019)

$$\int_{\mathbb{C}} p_n(z)\overline{z}^m \mathrm{e}^{-N|z|^2}|W(z)|^2 \mathrm{d}A(z) = \frac{1}{2\mathrm{i}} \int_\gamma p_n(z)\mu^{(m)}(z)\mathrm{d}z,$$

where

$$\mu^{(m)}(z) := W(z) \int_0^\infty s^m \overline{W(\overline{s})} e^{-Nzs} \, \mathrm{d}s$$

and γ *is a simple closed curve enclosing* $\{0, a_1, \ldots, a_\nu\}$ *counterclockwise.*

Theorem (Lee-Yang 2019)

Let $\vec{n} = (n_1, \cdots, n_\nu)$ *with non–negative integers* n_j*'s. We define* $p_{\vec{n}}(z)$ *to be the monic polynomial of degree* $|\vec{n}| = \sum_{j=1}^{\nu} n_j = n$ *satisfying the orthogonality conditions:*

$$\int_\gamma p_{\vec{n}}(z)\, z^k \mu_j(z) \mathrm{d}z = 0, \quad 0 \le k \le n_j - 1, 1 \le j \le \nu,$$

$$\mu_j(z) := W(z) \int_{\bar{a}_1}^{\infty} \prod_{k=1}^{\nu} (s - \bar{a}_k)^{n_k - \delta_{kj}} \overline{W(\bar{s})} e^{-Nzs}\, \mathrm{d}s.$$

Then

$$p_{\vec{n}}(z) = p_n(z), \quad n_j = \begin{cases} \kappa + 1, & j \le n - \kappa\nu, \\ \kappa, & \textit{otherwise.} \end{cases}$$

Here $\kappa := \lfloor n/\nu \rfloor$.

Planar O. P.=Type I M.O.P. Kuijlaars-Berezin-Parra 2023.

Let us define

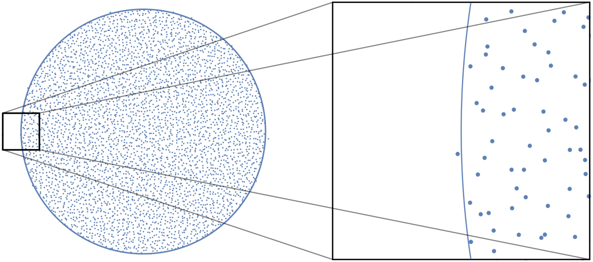
$$\psi(z) = [\mu_1(z), \ldots, \mu_\nu(z)].$$

The Riemann-Hilbert problem:

$$\begin{cases} Y_+(z) = Y_-(z) \begin{bmatrix} 1 & \psi(z) \\ 0 & I_\nu \end{bmatrix}, & \text{on } \bigcup_j \Gamma_{j0}, \\ Y_+(z) = Y_-(z) \begin{bmatrix} 1 & \psi(z) \\ 0 & I_\nu \end{bmatrix}_-^{-1} \begin{bmatrix} 1 & \psi(z) \\ 0 & I_\nu \end{bmatrix}_+, & \text{on } \mathbf{B} \cap (\Omega_0)^c, \\ Y(z) = \left(I_{\nu+1} + \mathcal{O}\left(\frac{1}{z}\right)\right) \cdot \operatorname{diag}\left(z^n, z^{-n_1}, \ldots, z^{-n_\nu}\right), & \text{as } z \to \infty, \\ Y \text{ is holomorphic matrix function}, & \text{otherwise.} \end{cases}$$

We have

$$[Y(z)]_{11} = p_n(z).$$



k-point correlation function

$$R_k(z_1, \ldots, z_k) := \frac{n!}{(n-k)!} \int_{\mathbb{C}^{n-k}} \mathbf{P}_n \prod_{j=k+1}^{n} \mathrm{d}A(z_j),$$

where

$$\mathbf{P}_n = \frac{1}{\mathcal{Z}_n} \prod_{i<j} |z_i - z_j|^2 \cdot \exp\Big(-n \sum_{j=1}^{n} Q(z_j) \Big).$$

Correlation Function

$$R_k(z_1, \ldots, z_k) = \det \left[\mathbf{K}_n(z_i, z_j)\right]_{i,j=1}^{k},$$

where $\mathbf{K}_n(z, \zeta)$ is the correlation kernel given by

$$\mathbf{K}_n(z, \zeta) := \mathrm{e}^{-\frac{N}{2}Q(z) - \frac{N}{2}Q(\zeta)} \sum_{k=0}^{n-1} \frac{1}{h_k} p_k(z)\overline{p_k(\zeta)}.$$

Here $p_n(z)$ satisfies the orthogonality condition,

$$\int_{\mathbb{C}} p_n(z)\overline{p_m(z)} e^{-NQ(z)} \mathrm{d}A(z) = h_n \delta_{nm} \quad (n, m = 0, 1, 2, \cdots).$$

Correlation Kernel: Ginibre kernel

The external potential Q satisfies the general settings.
Bulk Universality: Let z_0 be in the bulk of the droplet, let

$$K_n(\xi,\eta) = \frac{1}{2n\Delta Q(z_0)}\mathbf{K}_n\left(z_0 + \frac{\xi}{\sqrt{2n\Delta Q(z_0)}}, z_0 + \frac{\eta}{\sqrt{2n\Delta Q(z_0)}}\right),$$

there exist cocycles $c_n(\xi,\eta)$ such that

$$\lim_{n\to\infty} c_n(\xi,\eta)K_n(\xi,\eta) = G(\xi,\eta).$$

The Ginibre kernel

$$G(\xi,\eta) = \mathrm{e}^{\xi\overline{\eta}-\frac{1}{2}(|\xi|^2+|\eta|^2)}.$$

Cocycles: $c_n(A,B) = g_n(A)/g_n(B)$ where g_n is continuous.
Ameur-Hedenmalm-Makarov 2011.

Correlation Kernel: Faddeeva Plasma kernel

Edge Universality: Let z_0 be on the boundary of the droplet, let

$$K_n(\xi,\eta) = \frac{1}{2n\Delta Q(z_0)} \mathbf{K}_n\left(z_0 + \frac{\xi}{\sqrt{2n\Delta Q(z_0)}}, z_0 + \frac{\eta}{\sqrt{2n\Delta Q(z_0)}}\right),$$

there exist cocycles $c_n(\xi,\eta)$ such that

$$\lim_{n\to\infty} c_n(\xi,\eta) K_n(\xi,\eta) = G(\xi,\eta)\mathrm{erfc}(\xi + \overline{\eta}).$$

where

$$\mathrm{erfc}(z) = \frac{1}{\sqrt{2\pi}} \int_z^\infty \mathrm{e}^{-t^2/2} \mathrm{d}t.$$

Hedenmalm-Wennman 2021.

Is there any criticality universality in 2D Coulomb Gases?

The universal behavior at such critical points was conjectured that similar behavior will show up in 2D Coulomb Gases.
Bettelheim, Agam, Zabrodin, Wiegmann, 2005, etc.

Criticality universality in 1D Coulomb Gases
Bleher-Its 2003, Claeys-Kuijlaars 2006, etc.

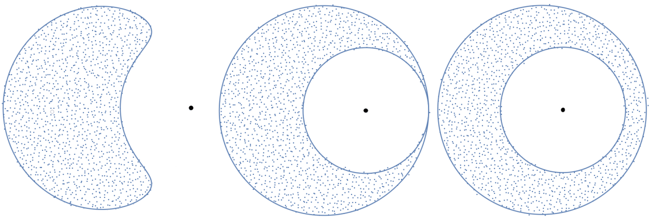
2D Painlevé II kernel

$$Q(z) = |z|^2 + 2c \log \frac{1}{|z-a|},$$

where $c > 0$ and $a \neq 0, \infty$.

Balogh-Bertola-Lee-McLaughlin 2015.

The droplet:



$$a > a_{cri}, \quad a \sim a_{cri} := \sqrt{c+1} - \sqrt{c}, \quad a < a_{cri}$$

2D Painlevé II kernel

Theorem (Krüger-Lee-Yang 2025)

Let z_0 be at the merging point, let

$$K_n(x,y,x',y') = \frac{1}{n^{5/6}}\mathbf{K}_n\Big(z_0 + \frac{x}{n^{1/2}} + \frac{\mathrm{i}y}{n^{1/3}}, z_0 + \frac{x'}{n^{1/2}} + \frac{\mathrm{i}y'}{n^{1/3}}\Big),$$

There exist cocycles $c_n(y,y')$ such that

$$\lim_{n\to\infty} c_n(y,y')K_n(x,y,x',y') = \mathbf{K}_s(x,y,x',y'),$$

where

$$\mathbf{K}_s(x,y,x',y') := \frac{\mathrm{e}^{-x^2-(x')^2}}{\sqrt{\pi/2}} \frac{\Psi_{21}(y;s)\Psi_{11}(y';s) - \Psi_{11}(y;s)\Psi_{21}(y';s)}{2\pi\mathrm{i}(y-y')},$$

where $\{\Psi_{jk}\}$ are related to Painlevé II equation.

Generalized Christoffel-Darboux identity

Let

$$\psi_n(z) := (z-a)^{Nc} p_n(z),$$

we write the pre-kernel as

$$\mathcal{K}_n(z,\zeta) := \mathrm{e}^{-Nz\bar{\zeta}} \sum_{k=0}^{n-1} \frac{1}{h_k} \psi_k(z) \overline{\psi_k(\zeta)}.$$

The correlation kernel can be written as

$$\mathbf{K}_n(z,\zeta) = \mathrm{e}^{-\frac{N}{2}|z|^2 - \frac{N}{2}|\zeta|^2 + Nz\bar{\zeta}} \frac{|z-a|^{Nc}|\zeta-a|^{Nc}}{(z-a)^{Nc}(\bar{\zeta}-a)^{Nc}} \mathcal{K}_n(z,\zeta).$$

Generalized Christoffel-Darboux identity

Theorem (Byun-Lee-Yang, arXiv:2107.07221)

Suppose that $a \neq 0$. Then we have the following form of the Christoffel-Darboux identity:

$$\overline{\partial}_\zeta \mathcal{K}_n(z,\zeta) = e^{-Nz\bar{\zeta}} \frac{1}{\frac{n+Nc}{N} h_{n-1} - h_n} \overline{\psi_n'(\zeta)} \Big(\psi_n(z) - z\psi_{n-1}(z)\Big)$$

$$- e^{-Nz\bar{\zeta}} \frac{p_{n+1}(a)}{p_n(a)} \frac{N\, h_n/h_{n-1}}{\frac{n+Nc+1}{N} h_n - h_{n+1}} \overline{\psi_{n-1}(\zeta)} \Big(\psi_{n+1}(z) - z\psi_n(z)\Big).$$

Partition Functions: Predictions

Given the external potential Q, the partition function:

$$Z_n^Q := \int_{\mathbb{C}^n} \prod_{i<j} |z_i - z_j|^2 \prod_{j=1}^{n} e^{-n\,Q(z_j)}\,\mathrm{d}A(z_j).$$

It is conjectured that if the droplet is *connected*, as $n \to \infty$, the partition function Z_n^Q has the asymptotic expansion of the form

$$\log Z_n^Q = C_1 n^2 + C_2 n \log n + C_3 n + C_4 \log n + C_5 + o(1).$$

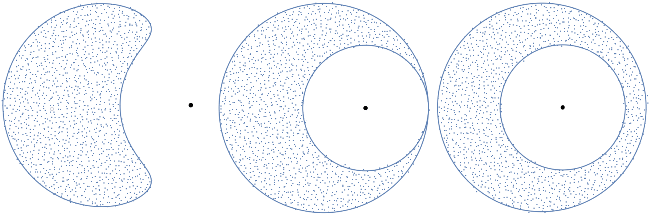
Ginibre Ensemble with Insertions

We consider the external potential,

$$Q(z) = |z|^2 + 2c \log \frac{1}{|z-a|},$$

where $c > 0$ and $a > 0$. The probability distribution is given by

$$\frac{1}{Z_n(a,c)} \prod_{i<j} |z_i - z_j|^2 \prod_{j=1}^{n} |z_j - a|^{2nc} \mathrm{e}^{-n|z_j|^2} \mathrm{d}A(z_j).$$



$$a > a_{cri}, \quad a \sim a_{cri} := \sqrt{c+1} - \sqrt{c}, \quad a < a_{cri}$$

Balogh-Bertola-Lee-McLaughlin 2015

Theorem (Byun-Seo-Yang 2025)

As $n \to \infty$, we have

$$\log Z_n(a,c) = -I_Q[\sigma_Q]n^2 + \frac{1}{2} n \log n + \Big(\frac{\log(2\pi)}{2} - 1\Big) n + \frac{6-\chi}{12}\log n + \frac{\log(2\pi)}{2} + \chi\,\zeta'(-1) + \mathcal{F}(a,c) + \mathcal{E}_n,$$

where

$$I_Q[\mu_Q] = \begin{cases} \frac{3}{4} + \frac{3c}{2} + \frac{c^2}{2}\log c - \frac{(c+1)^2}{2}\log(c+1) - ca^2, & a < a_{cri}, \\ \frac{3}{8} + \frac{a^2}{8} + \frac{3}{8a^2q^4} - \frac{5}{8q^2} + \Big(\frac{3}{4} + \frac{a^2}{8}\Big)a^2q^2 - \frac{3a^4q^4}{8} & \\ \quad + \log(2aq) + 2c\log(2aq^2) + \log\frac{(1+a^2q^2-2a^2q^4)^{c^2}}{(1+a^2q^2)^{(c+1)^2}}, & a > a_{cri}, \end{cases}$$

$q = q(a)$ is given explicitly,

Theorem (To be continued)

$$\chi = \begin{cases} 0, & a < a_{cri}, \\ 1, & a > a_{cri}, \end{cases}$$

$$\mathcal{F}(a,c) = \begin{cases} \frac{1}{12}\log\left(\frac{c}{1+c}\right), & a < a_{cri}, \\ \frac{1}{24}\log\left(\frac{(1+a^2q^2-2a^2q^4)^4}{(1+a^2q^2)^4(1-q^2)^3(1-a^4q^6)}\right), & a > a_{cri}, \end{cases}$$

and

$$\mathcal{E}_n = \begin{cases} \sum_{k=1}^{M}\left(\frac{B_{2k}}{2k(2k-1)}\frac{1}{n^{2k-1}} + \frac{B_{2k+2}}{4k(k+1)}\left(\frac{1}{(c+1)^{2k}} - \frac{1}{c^{2k}}\right)\frac{1}{n^{2k}}\right) \\ \quad +O(\frac{1}{n^{2M+1}}), & a < a_{cri}, \\ O\left(\frac{1}{n}\right), & a > a_{cri}, \end{cases}$$

for any $M > 0$, where $\{B_k\}$ are the Bernoulli numbers.

Theorem (Byun-Seo-Yang 2025)

For a fixed $c > 0$, let

$$a := a_{cri} - \frac{(\sqrt{c+1} - \sqrt{c})^{1/3}\mathbf{s}}{2(c^2+c)^{1/6}n^{2/3}} + O\Big(\frac{1}{n^{4/3}}\Big).$$

Then as $n \to \infty$, we have

$$\begin{aligned}\log Z_n(a,c) = & -\Big(\frac{3}{4} + \frac{3c}{2} + \frac{c^2}{2}\log c - \frac{(c+1)^2}{2}\log(c+1) - ca^2\Big)n^2 \\ & + \frac{1}{2}n\log n + \Big(\frac{\log(2\pi)}{2} - 1\Big)n + \frac{1}{2}\log n \\ & + \frac{\log(2\pi)}{2} + \frac{1}{12}\log\Big(\frac{c}{1+c}\Big) + \log F_{\mathrm{TW}}(c^{-2/3}\mathbf{s}) + O\Big(\frac{1}{n^{2/3}}\Big),\end{aligned}$$

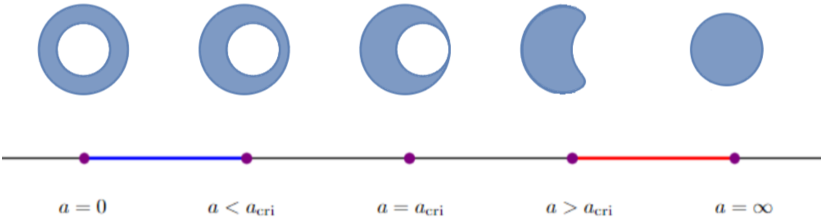
where

$$F_{\mathrm{TW}}(t) := \exp\Big(-\int_t^\infty (x-t)\mathbf{q}(x)^2 \mathrm{d}x\Big)$$

is the Tracy-Widom distribution.

Strategy of the Proof

- Deformation of the partition function
- Fine asymptotic behavior of orthogonal polynomials via Riemann-Hilbert problems.



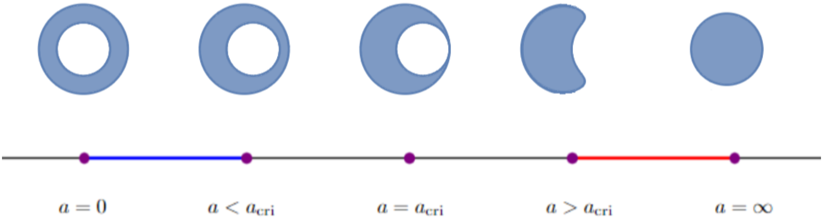
For $a < a_{cri}$:

$$\log Z_n(a,c) = \log Z_n(0,c) + \int_0^a \frac{d}{dt} \log Z_n(t,c)dt.$$

where

$$Z_n(0,c) = \frac{G(N+Nc+1)}{G(Nc+1)} \frac{N!}{N^{N^2c+N(N+1)/2}}.$$

Strategy of the Proof



For $a > a_{cri}$:

$$\log Z_n(a,c) = \log Z_n^{\rm Gin} + (2c\log a)n^2 - \int_a^\infty \left(\frac{d}{dt}\log Z_n(t,c) - \frac{2cn^2}{t}\right)dt.$$

where

$$Z_n^{\rm Gin} = \frac{N!G(N+1)}{N^{N(N+1)/2}}.$$

Critical Case: Duality. **Nishigaki-Kamenev 2002, Forrester-Rains 2009, Forrester 2025**

Ginibre Ensemble with (small) Insertions

We consider the external potential,

$$Q(z) = |z|^2 + 2\sum_{j=1}^{\nu} \frac{c_j}{n} \log \frac{1}{|z - a_j|},$$

where $\{c_1, \ldots, c_\nu\}$ are nonzero real numbers greater than -1 and $\{a_1, \ldots, a_\nu\}$ are distinct points in $\mathbb{D} \setminus \{0\}$. The probability distribution is given by

$$\frac{1}{Z_n} \prod_{i<j} |z_i - z_j|^2 \prod_{j=1}^{n} \prod_{k=1}^{\nu} |z_j - a_k|^{2c_k} \mathrm{e}^{-n|z_j|^2} \mathrm{d}A(z_j).$$

Partition Functions

Theorem (Lee-Yang 2025+)

If $\{a_i\}_{i=1}^{\nu}$ *are isolated, we have*

$$\mathcal{Z}_n = Const.\left(1 + \mathcal{O}\left(\frac{1}{N^\infty}\right)\right) \prod_{j=1}^{\nu} \mathrm{e}^{c_j N |a_j|^2} \prod_{i<j} |a_i - a_j|^{-2c_i c_j}.$$

If a_j *and* a_k *are merging, we have*

$$\mathcal{Z}_n = Const.\left(1 + \mathcal{O}\left(\frac{1}{N^\infty}\right)\right) \prod_{j=1}^{\nu} \mathrm{e}^{c_j N |a_j|^2} \prod_{i<j} |a_i - a_j|^{-2c_i c_j} \mathbf{F}\left(N|a_2 - a_1|^2\right),$$

where $\mathbf{F}$ *is related to Painlevé V.*

Webb-Wong 2018, Deaño-Simm 2019.
Remark: The isolated case. Bourgade, Dubach, Hartung, Keles 2025+.

Let us define the moments,

$$\nu_{jk}^{(i)} := \frac{1}{2\mathrm{i}} \int_{\gamma} z^{j+k} \, \mu_i(z) \, \mathrm{d}z, \quad \mu_{jk} := \int_{\mathbb{C}} z^j \, \bar{z}^k \, \mathrm{e}^{-N|z|^2} |W(z)|^2 \, \mathrm{d}A(z).$$

Let the n by n matrices of moments d_n and D_n be

$$D_n = \begin{bmatrix} \mu_{0,0} & \mu_{1,0} & \cdots & \mu_{n-1,0} \\ \mu_{0,1} & \mu_{1,1} & \cdots & \mu_{n-1,1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{0,n-1} & \mu_{1,n-1} & \cdots & \mu_{n-1,n-1} \end{bmatrix},$$

$$d_n = \begin{bmatrix} d_n^{(1)} \\ \vdots \\ d_n^{(\nu)} \end{bmatrix}, \quad d_n^{(j)} = \begin{bmatrix} \nu_{0,0}^{(j)} & \nu_{1,0}^{(j)} & \cdots & \nu_{n-1,0}^{(j)} \\ \vdots & \vdots & \vdots & \vdots \\ \nu_{0,n_j-1}^{(j)} & \nu_{1,n_j-1}^{(j)} & \cdots & \nu_{n-1,n_j-1}^{(j)} \end{bmatrix}.$$

Theorem (Lee-Yang 2019)

There exists a unique constant matrix A_n such that

$$d_n = A_n D_n$$

Theorem (Lee-Yang 2025+)

Let d_n be defined above, we have

$$\partial_{\bar{a}_1} \log\det d_n = \sum_{i\neq 1} \frac{n_1(n_i+c_i)+(n_1+c_1)n_i}{\bar{a}_1-\bar{a}_i} - Na_{1,1}^{(\mathbf{n})} - \sum_{j=n-\nu\kappa+1}^{\nu} \frac{a_{1,j}^{(\mathbf{n})}(n_j+c_j)}{\bar{a}_1-\bar{a}_j}.$$

where $a_{1,j}^{(\mathbf{n})}$ is the coefficient in the large z expansion of $z^{n_j}[Y_{\mathbf{n}}(z)]_{2(j+1)}$ as below.

$$z^{n_1}[Y_{\mathbf{n}}(z)]_{22} = 1 + \frac{a_{1,1}^{(\mathbf{n})}}{z} + \mathcal{O}\left(\frac{1}{z^2}\right),$$

$$z^{n_j}[Y_{\mathbf{n}}(z)]_{2(j+1)} = \frac{a_{1,j}^{(\mathbf{n})}}{z} + \mathcal{O}\left(\frac{1}{z^2}\right), \quad j>1.$$

Happy Birthday, Peter!
Thank You For Your Attention!