Random winding numbers

associated to determinantal curves over $\mathbb{S}^1$ for chiral Hamiltonians

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Log-gases in Caeli Australi 2025, in honor of Peter Forrester

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THE UNIVERSITY OF MELBOURNE

A short story about topology and matrices

A footbridge between RMT and Topology

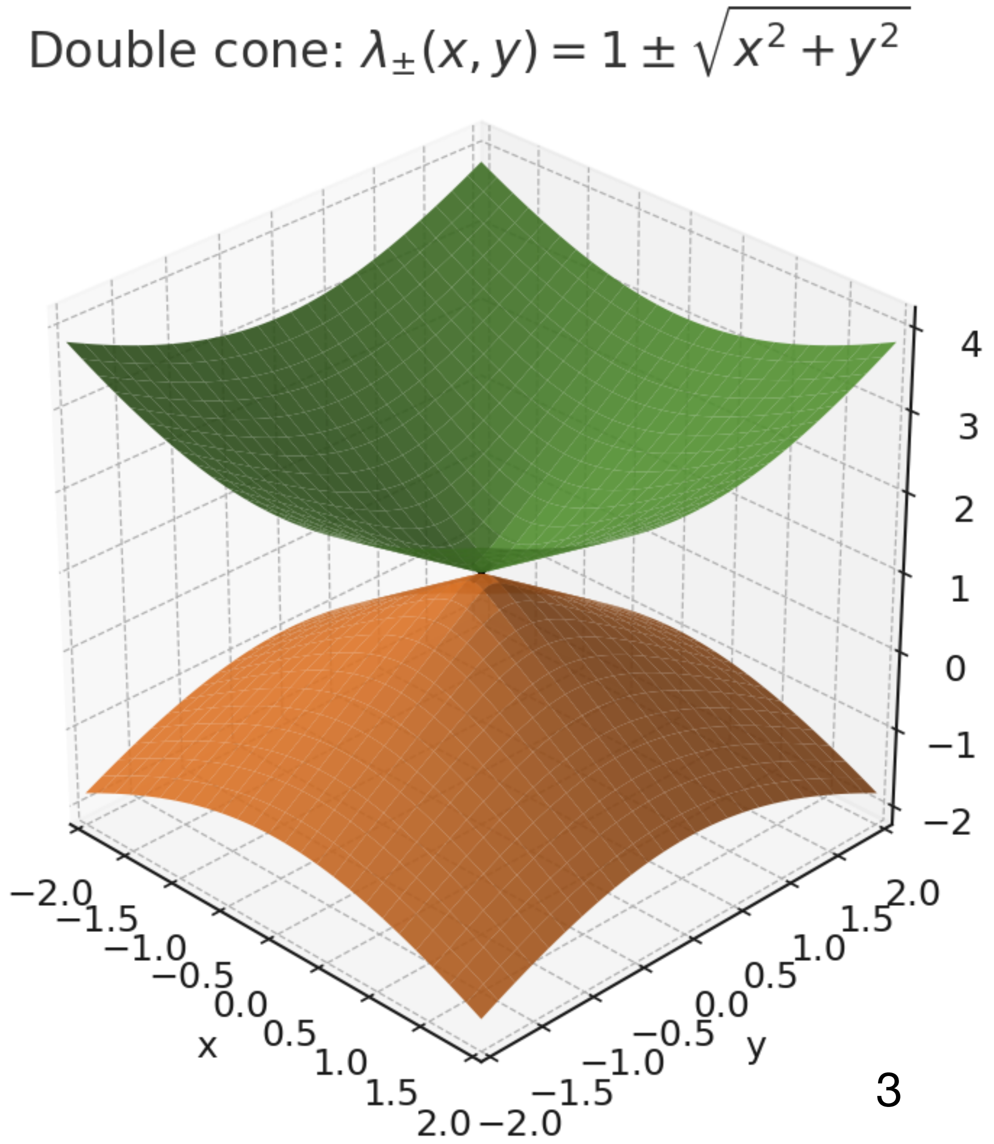
Based on Peter's notes distributed at the Melbourne Uni RMT Seminar in 2022

- We consider: $A : \mathbb{R}_+ \times [0,2\pi) \longrightarrow \mathcal{S}_2(\mathbb{R})$

$$(r,\theta) \mapsto I_2 + r \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix}$$

- Two eigenvalues $\lambda_\pm(r,\theta) = 1 \pm r$, with a degeneracy at $r = 0$

- In Cartesian coordinates:



- Hamiltonian for a spin in a magnetic field along the z-axis, intensity driven by r and modulated by $\cos(\theta)$

- Eigenspaces $\mathrm{E}_\pm(\theta) \in \mathbb{R}\mathrm{P}^1$ are spanned by the two orthonormal vectors:

$$v_-(\theta) = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) \end{pmatrix}, v_+(\theta) = \begin{pmatrix} -\sin\left(\frac{\theta}{2}\right) \\ \cos\left(\frac{\theta}{2}\right) \end{pmatrix}.$$

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- Something remarkable happens:

- $A(r, \cdot\,)$ is 2π-periodic, the eigenvalues $\lambda_{\pm}(\theta)$ are also 2π-periodic.

- The eigenvectors $v_{\pm}(\theta)$ are 4π-periodic !

- The non-2π-periodicity reflects a topological obstruction

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- Could we have changed the coefficients of A so that the maps $\theta \mapsto v_{\pm}(\theta)$ are:
    1. 6π, 8π… more generally speaking $2\pi n$-periodic for $n \in \mathbb{N}^*$?
    2. T-periodic for an arbitrary $T > 0$?
    3. Aperiodic ?

- What if the coefficients of A were complex-valued functions ?

A footbridge between RMT and Topology

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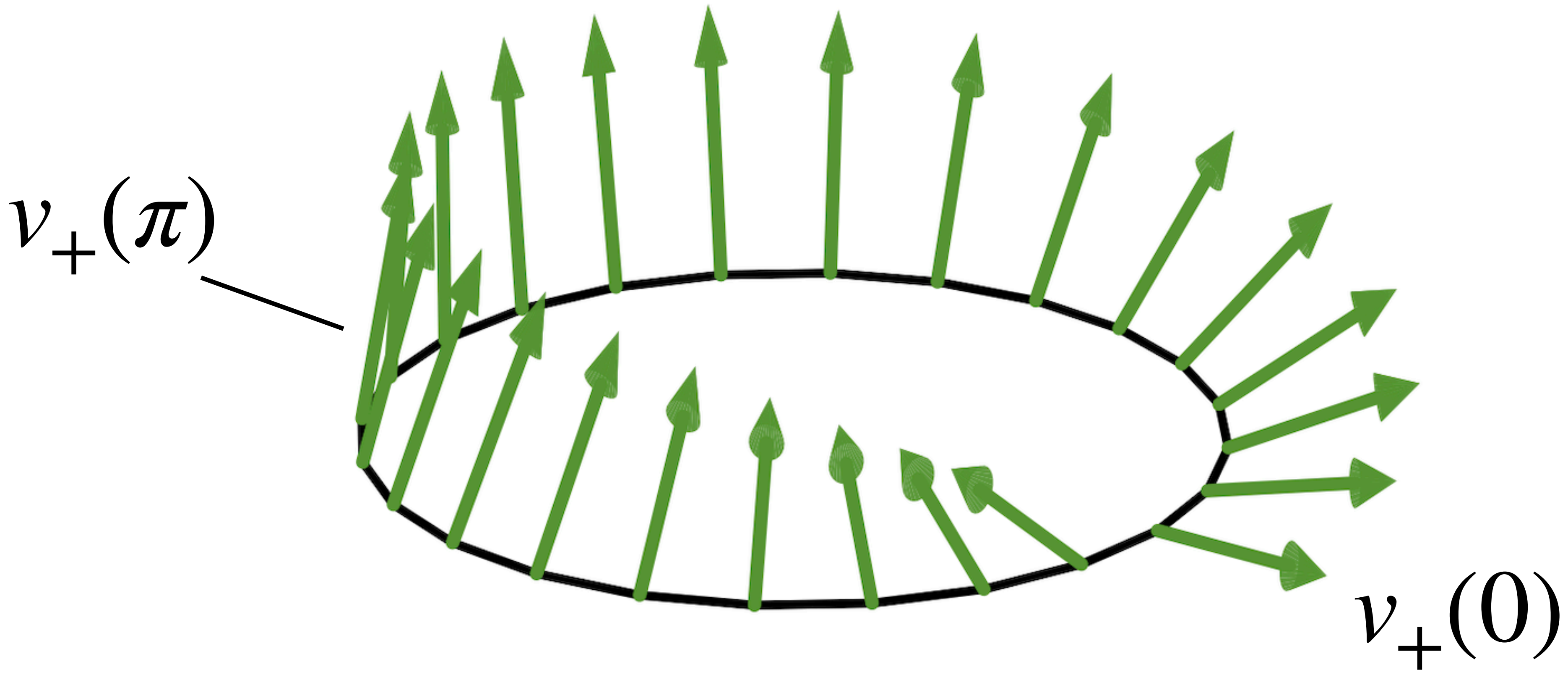
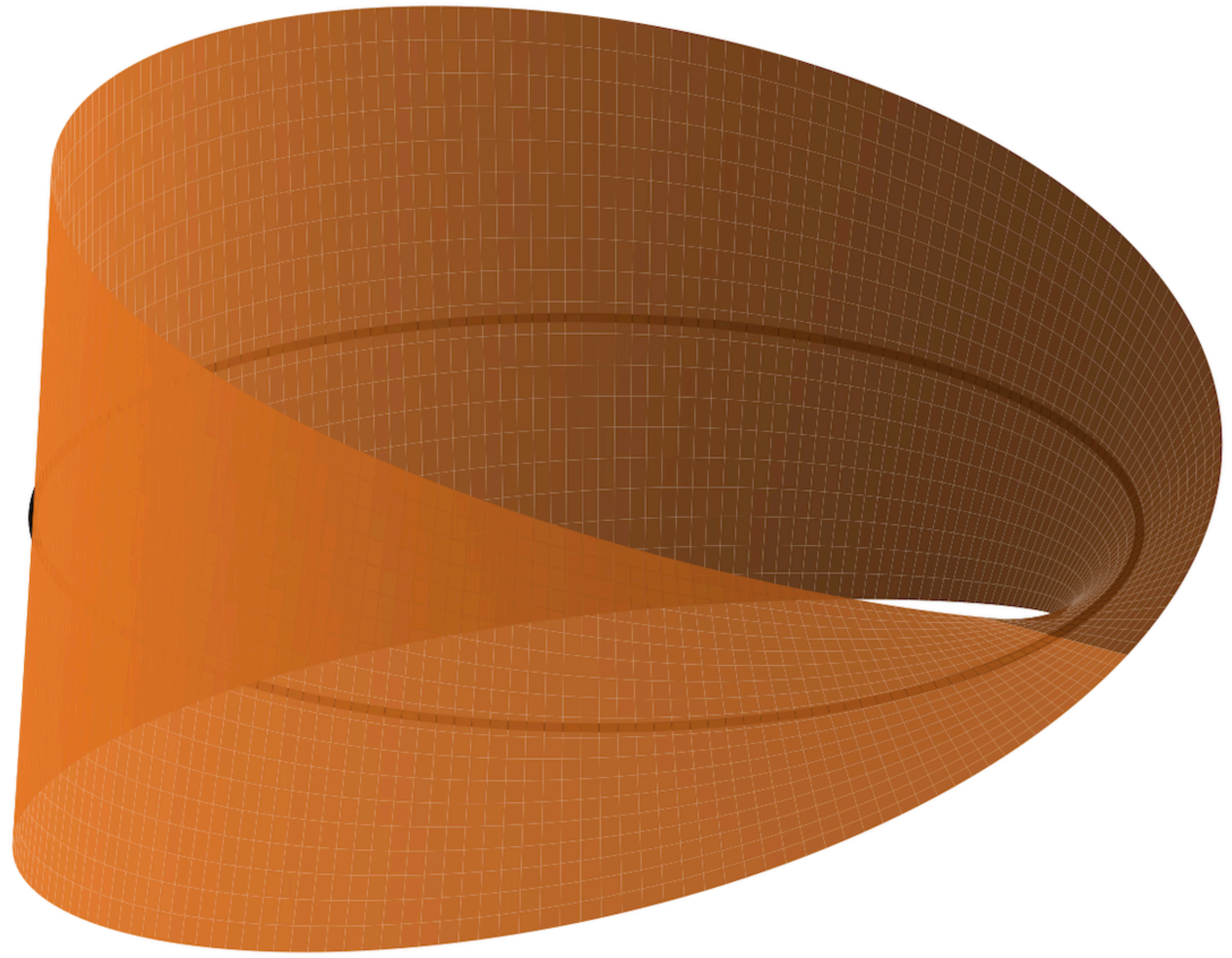
- A different perspective: the maps $\theta \mapsto v_{\pm}(\theta)$ form line bundles over S^1





- The obstruction we observed before corresponds to the non-orientability of the Möbius strip
- Real line bundles over S^1 are classified by the first Stiefel-Whitney class $w_1(S^1) \in \mathrm{H}^1\left(\mathrm{S}^1, \frac{\mathbb{Z}}{2\mathbb{Z}}\right) \cong \frac{\mathbb{Z}}{2\mathbb{Z}}$

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- There are only two possibilities : either $v_\pm(\theta + 2\pi) = v_\pm(\theta)$ or $v_\pm(\theta + 2\pi) = -v_\pm(\theta)$
  it rules out T-periodicity for a non-integer T and aperiodicity !

- Complex line bundles over S^1 are classified by the first Chern class $c_1(S^1) \in \mathrm{H}^2(\mathrm{S}^1, \mathbb{Z}) \cong \{\mathrm{e}\}$
- All complex line bundles over S^1 are trivial: non-zero sections can be "gauged" by multiplication with a phase function to have any periodicity or to be aperiodic.

Physics motivation

Physics motivation

Disordered quantum 1D chiral systems with edges

- Goal: Classification of topological phases for 1D quantum systems in the class AIII.
- Hamiltonian has the following form:

$$H(p) = \begin{pmatrix} 0_N & K(p) \\ K(p)^* & 0_N \end{pmatrix}$$

- $p \in \mathbb{S}^1$ is the crystal momentum
- $K(p)$ is a $N \times N$ matrix with complex entries
- H classified through the winding number of

$$p \mapsto \det\left(H(p)\right)$$

Altland-Zirnbauer classification

Symmetry				Dimension							
AZ	T	C	S	1	2	3	4	5	6	7	8
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

courtesy to J. Baez, 2010.

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- Matrix representation of the QCD Dirac Operator at chemical potential $\mu = 0$ mentioned earlier by Jacobus

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- Disordered system implies $K(p)$ is a random matrix. Set $\mathscr{C} : p \mapsto \det\left(K(p)\right)$.
- Mathematical quantity we study:

$$\mathrm{Wind}_{\mathrm{N}}(\mathscr{C},0) = \frac{1}{2\pi \mathrm{i}} \oint_{\mathbb{S}^1} \mathrm{w}(\mathrm{p})\,\mathrm{dp}$$

where w is the winding number density:

$$w(p) = \frac{\mathrm{d}}{\mathrm{dp}} \log\left(\det\left(K(p)\right)\right) = \frac{1}{\det\left(K(p)\right)} \frac{\mathrm{d}}{\mathrm{dp}} \det\left(K(p)\right)$$

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- Associated partition function:

$$\mathscr{Z}_m^{(N)}\left(\mathbf{p},\mathbf{q}\right) = \mathbb{E}\left(\prod_{j=1}^{m} \frac{\det\left(K\left(p_j\right)\right)}{\det\left(K\left(q_j\right)\right)}\right)$$

- Allows to recover m-point correlation functions:

$$\mathrm{C}_1^{(\mathrm{N})}\left(\mathrm{p}\right) = \mathbb{E}\left(\mathrm{w}\left(\mathrm{p}\right)\right) = \frac{\mathrm{d}}{\mathrm{dq}} \mathscr{Z}_1^{(\mathrm{N})}\left(\mathrm{p},\mathrm{q}\right)\Big|_{\mathrm{p}=\mathrm{q}}$$

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- Initial model considered in 2021:

$$K(p)=\cos(p)G_1+\sin(p)G_2$$

where G_1, G_2 are idpt drawn from $\mathrm{GinUE(N)}$

Results:

$$C_1(p)=0, C_2(p_1,p_2)=-\frac{1-\cos(p_1-p_2)^{2N}}{1-\cos(p_1-p_2)^2}$$

Braun, Hahn, Waltner, Gat, Guhr,
“Winding Number Statistics of a Parametric Chiral Unitary Random Matrix Ensemble”, 2021.

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- A recent model considered in 2023:

$$K(p) = a(p)G_1 + b(p)G_2$$

where G_1, G_2 are idpt drawn from $\mathrm{GinUE(N)}$, (a,b) are two smooth $\mathbb{C}$-valued functions on S^1

Results:

$$\mathcal{Z}_m^{(N)}(\mathbf{p},\mathbf{q}) = \frac{\det\left(\frac{1}{\nu(p_i)^\top J\nu\left(q_j\right)}\left(\frac{\nu\left(q_j\right)^\dagger\nu(p_i)}{\nu\left(q_j\right)^\dagger\nu\left(q_j\right)}\right)^N\right)_{1\le i,j\le m}}{\det\left(\frac{1}{\nu(p_i)^\top J\nu\left(q_j\right)}\right)_{1\le i,j\le m}}$$

$$\nu(p) = \begin{pmatrix} a(p) \\ b(p) \end{pmatrix} \in \mathbb{C}^2, J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

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Hahn, Kieburg, Gat, Guhr, "Winding Number Statistics for Chiral Random Matrices: Averaging Ratios of Determinants with Parametric Dependence", 2023

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- An extension studied in late 2023:

$$K(p) = a(p)G_1 + b(p)G_2$$

where G_1, G_2 are idpt drawn from $\mathrm{GinOE(N)}$, (a, b) are two smooth $\mathbb{C}$-valued functions on S^1

Results:

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- For us today: Asymptotic expansion of

$$\mathbb{E}\left(\mathrm{Wind_N}\right)=\frac{1}{2\pi i}\oint_{\mathbb{S}^1}\mathbb{E}\left(w\left(p\right)\right)\mathrm{dp}$$

- K is a 2-matrix model:

$$K\left(p\right)=a\left(p\right)K_1+b\left(p\right)K_2$$

- a, b two smooth complex-valued functions on $\mathbb{S}^1$
- K_1, K_2 two iid random matrices w complex entries

Physics motivation

Example of determinantal curves : G_1, G_2 drawn from Ginibre Unitary Ensemble (GinUE)

$$\mathscr{C} : \theta \mapsto \det\left(\cos(\theta)\, G_1 + \sin(\theta)\, G_2\right)$$

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Random Matrix Theory

Random Matrix Theory

Pólya Ensembles of multiplicative type on $\mathrm{Gl}_{\mathrm{N}}(\mathbb{C})$

- A class of $N \times N$ random matrices with complex entries and isotropic eigenspectrum containing most well-known ensembles.

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- Key properties:
  1. Isotropic eigenspectrum: $\mathrm{d}\mathbb{P}\left(\mathrm{V}_1\mathrm{M}\mathrm{V}_2\right) = \mathrm{d}\mathbb{P}\left(\mathrm{M}\right)$ where $V_1, V_2 \in \mathrm{U}_\mathrm{N}$,
  2. Parametrised by a single function (*Pólya weight*) ω and denoted $X \sim \mathrm{Pól}_N[\omega]$,
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$$\check{\omega}(x) = x^{-(N+1)}\omega\left(\frac{1}{x}\right), \quad \left(\omega_1 \circledast \omega_2\right)(x) = \int_0^{+\infty} \omega_1\left(\frac{x}{y}\right)\omega_2(y)\frac{\mathrm{d}y}{y}.$$

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Pólya Ensembles of multiplicative type on $\mathrm{Gl}_N(\mathbb{C})$

- Example: Ginibre Unitary Ensemble (GinUE)

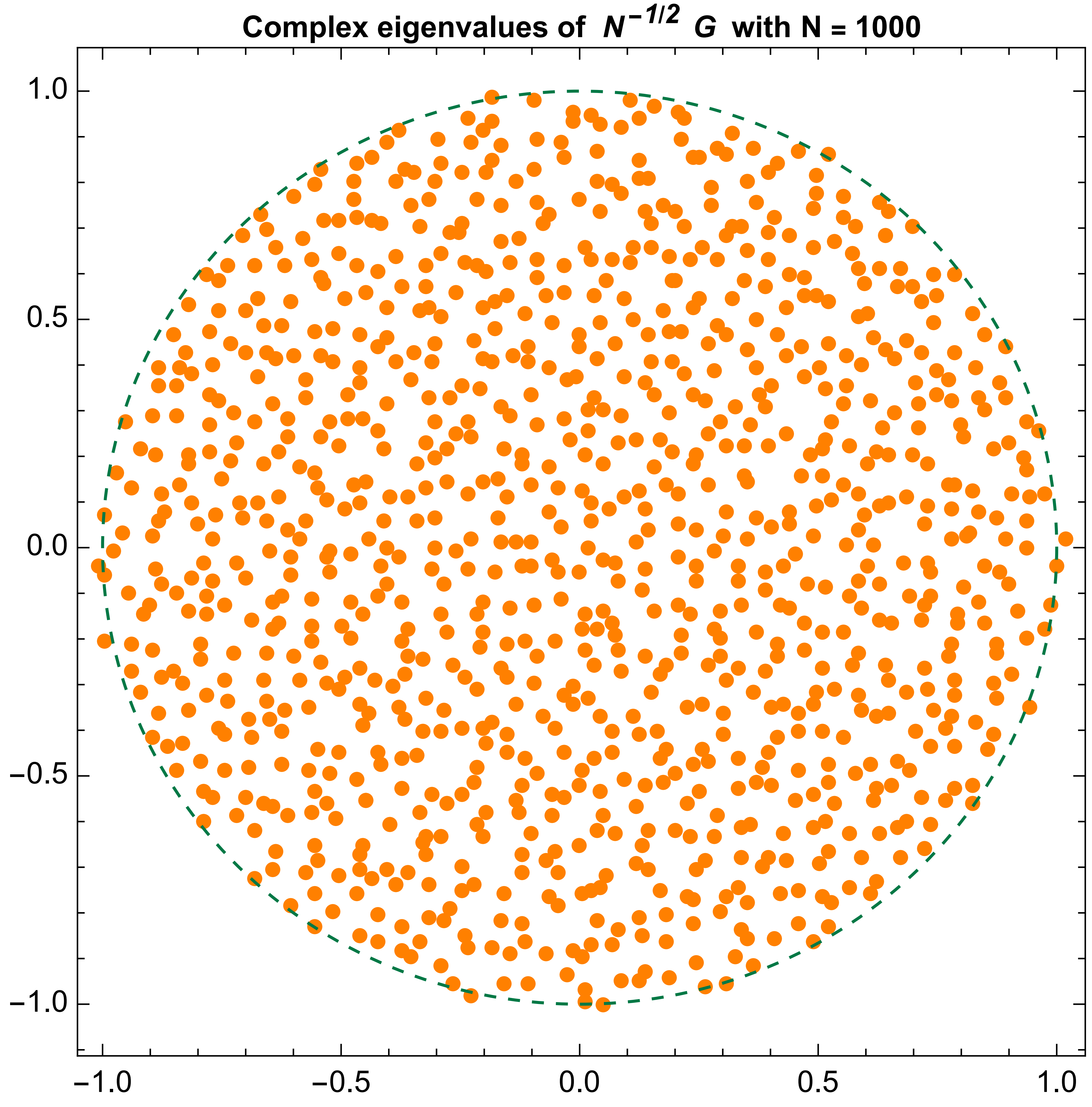
1. $G = \left(Z_{i,j}\right)$ a $N \times N$ random matrix with iid entries

2. $Z_{1,1} \sim \mathcal{N}_{\mathbb{C}}(0,1)$

3. $\mathrm{d}\mathbb{P}_G(\mathrm{M}) \propto \exp\left(-\mathrm{tr}\left(\mathrm{MM}^*\right)\right)\mathrm{dM}$

4. $\omega_{\mathrm{Gin}}(t) = e^{-t}$

5. $f_{\mathrm{EV}}\left(z_1, \ldots, z_N\right) \propto \left|\Delta_N(\mathbf{z})\right|^2 \prod_{k=1}^{N} e^{-\left|z_k\right|^2}$



$$\left|\Delta_N(\mathbf{z})\right|^2 = \prod_{1 \le i < j \le N} \left|z_j - z_i\right|^2$$

Forrester & Byun, "Progress on the study of the Ginibre ensembles I: GinUE", 2023.

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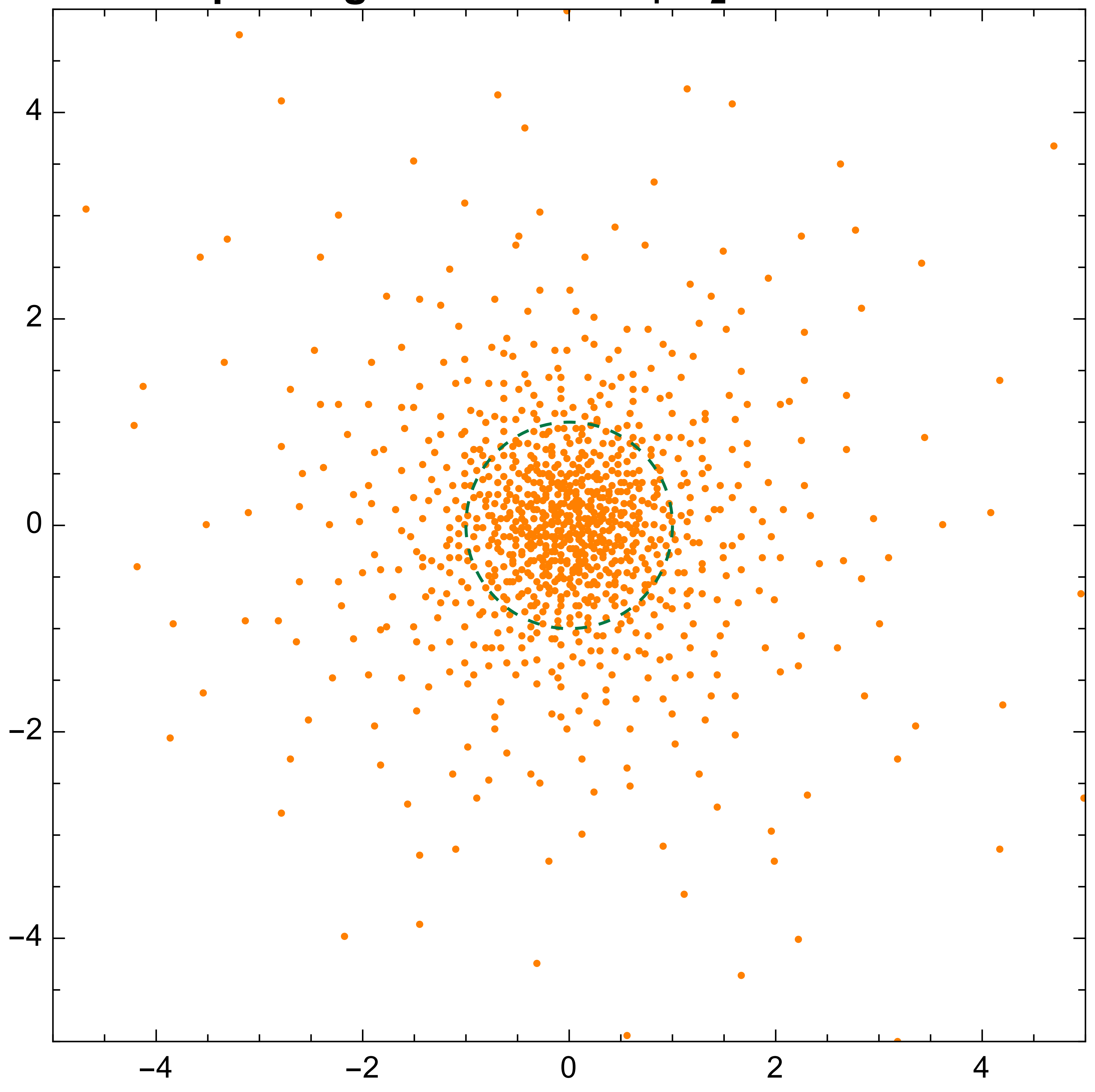
- Example: Complex Spherical Ensemble
  1. G_1, G_2 iid from GinUE of size N
  2. $S = G_1 G_2^{-1}$ the generalized ratio
  3. $\mathrm{d}\mathbb{P}_S(\mathrm{M}) \propto \det\left(\mathrm{I}_N + \mathrm{MM}^*\right)^{-2N} \mathrm{dM}$
  4. $\omega_{\mathrm{Sph}}(t) = N!(1+t)^{-(N+1)}$, $\omega_{\mathrm{Sph}} = \omega_{\mathrm{Gin}} \circledast \check{\omega}_{\mathrm{Gin}}$
  5. $f_{\mathrm{EV}}\left(z_1, \ldots, z_N\right) \propto \left|\Delta_N(\mathbf{z})\right|^2 \prod_{k=1}^{N} (1 + \left|z_k\right|^2)^{-(N+1)}$

Complex eigenvalues of $G_1\, G_2^{-1}$ for N = 1000



$$\left|\Delta_N(\mathbf{z})\right|^2 = \prod_{1 \le i < j \le N} \left|z_j - z_i\right|^2$$

Forrester & Byun, "Progress on the study of the Ginibre ensembles I: GinUE", 2023.

Random Matrix Theory

Pólya Ensembles of multiplicative type on $\mathrm{Gl}_{\mathrm{N}}(\mathbb{C})$

- ω is s.t $x \mapsto \widetilde{\omega}(x) = e^{-x}\omega\left(e^{-x}\right)$ is a Pólya Frequency Function of order N :

$$\forall k \in \{1,\ldots,N\} : \Delta_k(\mathbf{x})\,\Delta_k\left(\mathbf{y}\right)\det\left(\widetilde{\omega}\left(x_i - y_j\right)\right) \geq 0.$$

- Continuous integrable functions in PF_2 are exactly log-concave functions.

- Encapsulates many famous RMT ensembles : Wishart-Laguerre, Cauchy-Lorentz, Truncated Unitary Matrices, Muttalib-Borodin, Meijer-G Ensembles…

Kieburg & Kösters, "Exact relation between singular value and eigenvalue statistics", Random Matrices: Theory Appl., 2016.

Förster, Kieburg & Kösters, "Polynomial ensembles and Pólya frequency functions", J. Theor. Probab., 2020.

Results

Results

Partition function

- Our model : $K\left(p\right) = a\left(p\right)K_1 + b\left(p\right)K_2$ (2-matrix model)
- K_1 and K_2 iid with $K_1 \sim \text{Pól}_N[\omega]$, $a, b \in \mathscr{C}^2\left(\mathbb{S}^1, \mathbb{C}\right)$

- Partition function :

$$\mathscr{Z}_m^{(N)}\left(\mathbf{p}, \mathbf{q}\right) = \mathbb{E}\left(\prod_{j=1}^{m} \frac{\det\left(K\left(p_j\right)\right)}{\det\left(K\left(q_j\right)\right)}\right)$$

- 1-point correlation :

$$\mathrm{C}_1^{(\mathrm{N})}\left(\mathrm{p}\right) = \mathbb{E}\left(\mathrm{w}\left(\mathrm{p}\right)\right) = \frac{\mathrm{d}}{\mathrm{dq}}\mathscr{Z}_1^{(\mathrm{N})}\left(\mathrm{p}, \mathrm{q}\right)\Big|_{\mathrm{p}=\mathrm{q}}$$

Results

Partition function

- Our model : $K(p) = a(p)\, K_1 + b(p)\, K_2$ (2-matrix model)
- K_1 and K_2 iid with $K_1 \sim \text{Pól}_N[\omega]$, $a, b \in \mathscr{C}^2(\mathbb{S}^1, \mathbb{C})$
- Our strategy : reduce the 2-matrix model to 1-matrix one.

$$\mathbb{E}\left(\prod_{j=1}^{m} \frac{\det\left(a(p_j) K_1 + b(p_j) K_2\right)}{\det\left(a(q_j) K_1 + b(q_j) K_2\right)}\right) = \left[\prod_{i=1}^{m} \frac{b(p_i)}{b(q_i)}\right]^N \mathbb{E}\left(\prod_{j=1}^{m} \frac{\det\left(\kappa(p_j) I_N + K_1^{-1}K_2\right)}{\det\left(\kappa(q_j) I_N + K_1^{-1}K_2\right)}\right)$$

$$\kappa(p) = \frac{a(p)}{b(p)}, \qquad K_1^{-1}K_2 \sim \text{Pól}_N\left[\check{\omega} \circledast \omega\right]$$

- Partition function :

$$\mathscr{Z}_m^{(N)}(\mathbf{p}, \mathbf{q}) = \mathbb{E}\left(\prod_{j=1}^{m} \frac{\det\left(K(p_j)\right)}{\det\left(K(q_j)\right)}\right)$$

- 1-point correlation :

$$\mathrm{C}_1^{(\mathrm{N})}(\mathrm{p}) = \mathbb{E}\left(\mathrm{w}(\mathrm{p})\right) = \frac{\mathrm{d}}{\mathrm{dq}} \mathscr{Z}_1^{(\mathrm{N})}(\mathrm{p}, \mathrm{q})\Big|_{\mathrm{p}=\mathrm{q}}$$

Results

Partition function

- Formula for the Partition function:

$$\mathscr{Z}_m^{(N)}\left(\mathbf{p},\mathbf{q}\right) = \frac{\det\left(\widetilde{\mathrm{Q}}_m^{(N)}[\omega](\mathbf{p},\mathbf{q})\right)}{\det\left(\mathrm{Q}_m\left(\mathbf{p},\mathbf{q}\right)\right)}$$

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where we set $\nu\left(p\right) = \begin{pmatrix} a\left(p\right) \\ b\left(p\right) \end{pmatrix} \in \mathbb{C}^2, J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\Upsilon_N\left(z_1,z_2\right) = \sum_{k=1}^{N} \frac{\mathscr{M}\left[\check{\omega}\circledast\omega\right]\left(k,\left|z_2\right|^2\right)}{\mathscr{M}\left[\check{\omega}\circledast\omega\right]\left(k\right)}\left(\frac{z_1}{z_2}\right)^k$

- The kernels Q_{m} and $\widetilde{\mathrm{Q}}_m^{(N)}[\omega]$ are :

$$\mathrm{Q}_{\mathrm{m}}(\mathbf{p},\mathbf{q}) = \left(\frac{1}{\nu\left(\mathrm{p_k}\right)^\top \mathrm{J}\nu\left(\mathrm{q_j}\right)}\right)_{1\le \mathrm{k,j} \le \mathrm{m}}, \quad \widetilde{\mathrm{Q}}_m^{(N)}[\omega](\mathbf{p},\mathbf{q}) = \left(\frac{\left(b(p_k)/b(q_j)\right)^N}{\nu(p_k)^\top \mathrm{J}\nu(q_j)}\left[1+\left(1-\frac{\kappa\left(q_j\right)}{\kappa\left(p_k\right)}\right)\Upsilon_N\big(\kappa(p_k),\kappa(q_j)\big)\right]\right)_{1\le k,j\le m}$$

Results

Partition function

- Formula for the Partition function:

$$\mathcal{Z}_m^{(N)}(\mathbf{p},\mathbf{q}) = \frac{\det\left(\widetilde{\mathrm{Q}}_m^{(N)}[\omega](\mathbf{p},\mathbf{q})\right)}{\det\left(\mathrm{Q}_m(\mathbf{p},\mathbf{q})\right)}$$

where we set $\nu(p) = \begin{pmatrix} a(p) \\ b(p) \end{pmatrix} \in \mathbb{C}^2, J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\Upsilon_N(z_1,z_2) = \sum_{k=1}^{N} \frac{\mathcal{M}[\check{\omega} \circledast \omega]\left(k, |z_2|^2\right)}{\mathcal{M}[\check{\omega} \circledast \omega](k)} \left(\frac{z_1}{z_2}\right)^k$

Mellin transformations

$$\mathcal{M}[f](z) \mapsto \int_0^{+\infty} t^{z-1} f(t)\,\mathrm{dt}$$

$$\mathcal{M}[f](z,A) \int_0^{A} t^{z-1} f(t)\,\mathrm{dt}$$

- The kernels $\mathrm{Q_m}$ and $\widetilde{\mathrm{Q}}_m^{(N)}[\omega]$ are :

$$\mathrm{Q_m}(\mathbf{p},\mathbf{q}) = \left(\frac{1}{\nu(\mathrm{p_k})^{\top} \mathrm{J}\nu(\mathrm{q_j})}\right)_{1\le k,j\le m},\quad \widetilde{\mathrm{Q}}_m^{(N)}[\omega](\mathbf{p},\mathbf{q}) = \left(\frac{\left(b(p_k)/b(q_j)\right)^N}{\nu(p_k)^{\top}\mathrm{J}\nu(q_j)}\left[1 + \left(1 - \frac{\kappa(q_j)}{\kappa(p_k)}\right)\Upsilon_N(\kappa(p_k),\kappa(q_j))\right]\right)_{1\le k,j\le m}$$

Results

Average winding number

- Large enough class of Pólya weight with exponential decay : $\omega(t) = t^{\delta} e^{-t^{\gamma}}, \delta > -1, \gamma > 0.$

$$\mathbb{E}\left(\mathrm{Wind_N}\right) = \frac{N}{\gamma} \oint_{\mathbb{S}^1} \frac{\nu_\gamma(p)^* \nu_\gamma'(p)}{\left\| \nu_\gamma(p) \right\|^2} \frac{\mathrm{dp}}{2\pi i} + \frac{\gamma - 1}{2} \oint_{\mathbb{S}^1} \frac{\kappa'(p)}{\kappa(p)} \left[\frac{\left\| a(p) \right\|^{2\gamma} - \left\| b(p) \right\|^{2\gamma}}{\left\| a(p) \right\|^{2\gamma} + \left\| b(p) \right\|^{2\gamma}} \right] \frac{\mathrm{dp}}{2\pi i} + o(1)$$

where $\nu_\gamma(p) = \begin{pmatrix} a(p)^\gamma \\ b(p)^\gamma \end{pmatrix}$

Results

Average winding number

- Large enough class of Pólya weight with exponential decay : $\omega(t) = t^{\delta} e^{-t^{\gamma}}, \delta > -1, \gamma \in \mathbb{N}^*$.

$$\mathbb{E}\left(\mathrm{Wind}_{\mathrm{N}}\right) = \frac{N}{\gamma} \oint_{\mathbb{S}^1} \frac{\nu_\gamma(p)^* \nu_\gamma'(p)}{\| \nu_\gamma(p) \|^2} \frac{\mathrm{dp}}{2\pi i} + \frac{\gamma - 1}{2} \oint_{\mathbb{S}^1} \frac{\kappa'(p)}{\kappa(p)} \left[\frac{\| a(p) \|^{2\gamma} - \| b(p) \|^{2\gamma}}{\| a(p) \|^{2\gamma} + \| b(p) \|^{2\gamma}} \right] \frac{\mathrm{dp}}{2\pi i} + o(1)$$

where $\nu_\gamma(p) = \begin{pmatrix} a(p)^\gamma \\ b(p)^\gamma \end{pmatrix}$

Berry phase

non-Gaussian effects

Asymmetry on the parameter functions of the model

What's next ?

Variance winding number

- Pólya weights with bounded support.
- Variance of the Winding Number (work in progress)
- Central Limit Theorem for the Winding Number (work in progress)
- General $\mathrm{GinUE(N)}$-valued Random Fields on S^1 beyond the 2-matrix model (work in progress)

$$K : S^1 \mapsto \mathrm{M_N}(\mathbb{C}) \text{ such that: } \mathbb{E}(K(p)_{i,j}\overline{K(q)_{k,\ell}}) = S(p,q)\delta_{i,k}\delta_{j,\ell}$$

$$\mathbb{E}(K(p)_{i,j}K(q)_{k,\ell}) = 0$$

- Investigating symmetry classes BDI and CII in 1D (corresponding to GinOE and GinSE)
- Investigating Random Chern numbers in higher dimensions