# Optimal decay of eigenvector overlap for non-Hermitian i.i.d matrices

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Log-gases in Caeli Australi:
Recent Developments in and around Random Matrix Theory

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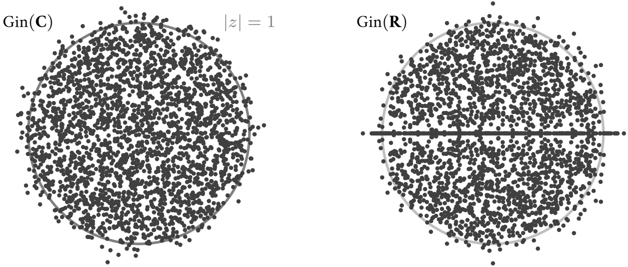
# Outline of the talk

# Wigner matrix v.s. i.i.d. matrix

| | Wigner matrix | i.i.d. matrix |
|---|---|---|
| Empirical spectral distribution | Semicircle law supported on [-2,2] | Circular law supported on the unit disk |
| complex Gaussian | Gaussian unitary ensemble (GUE) | complex Ginibre ensemble |
| Schur decomposition | $X = U^* \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_N \end{pmatrix} U$ | $X = U^* \begin{pmatrix} \lambda_1 & T_{12} & \cdots & T_{1N} \\ 0 & \lambda_2 & \cdots & T_{2N} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_N \end{pmatrix} U$ |
| Joint eigenvalue distribution | $\frac{1}{Z_N} \prod_{j<k} (\lambda_j - \lambda_k)^2 \prod_{k=1}^{N} e^{-N\lambda_k^2}, \ \lambda_k \in \mathbb{R}$ | $\frac{1}{Z_N} \prod_{j<k} \lvert\lambda_j - \lambda_k\rvert^2 \prod_{k=1}^{N} e^{-N\lvert\lambda_k\rvert^2}, \ \lambda_k \in \mathbb{C}$ |
| Eigenvector distribution | Haar distributed on U(N) independent of e.v. | different than U(N) correlated with e.v. |
| real Gaussian | Eigenvalue: β=1<br>Eigenvector: O(N) | more complicated |

# Complex v.s. Real Ginibre

- Many work on eigenvalues, see survey book [Forrester,Byun'25].



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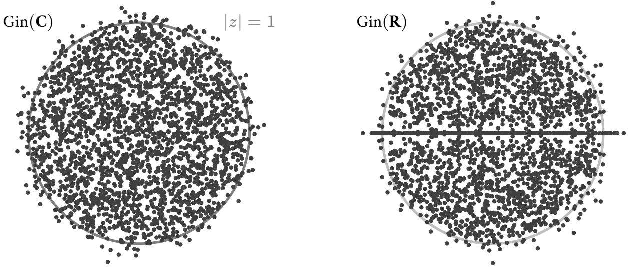



- Left/right eigenvectors form bi-orthogonal basis:

$$X\mathbf{r}_i = \lambda_i \mathbf{r}_i, \quad \mathbf{l}_i^* X = \lambda_i \mathbf{l}_i^*, \quad \langle \mathbf{l}_i, \mathbf{r}_j \rangle = \delta_{ij}.$$

- Define *eigenvector overlap* to quantify non-orthogonality:

$$\mathcal{O}_{ij} := \langle \mathbf{l}_i, \mathbf{l}_j \rangle \langle \mathbf{r}_j, \mathbf{r}_i \rangle, \quad \langle \mathbf{l}_i, \mathbf{r}_j \rangle = \delta_{ij}.$$

# Eigenvector overlaps

- $\mathcal{O}_{ij}$ is invariant under rescalings and $\sum_i \mathcal{O}_{ij} = 1$.

$$\mathcal{O}_{ij} := \langle \mathbf{l}_i, \mathbf{l}_j \rangle \langle \mathbf{r}_j, \mathbf{r}_i \rangle, \quad \langle \mathbf{l}_i, \mathbf{r}_j \rangle = \delta_{ij}$$

- $\sqrt{\mathcal{O}_{ii}}$ is also known as *condition number* in smoothed analysis, describing how sensitive the eigenvalue is to small perturbations:

$$\sqrt{\mathcal{O}_{ii}} = \lim_{t \to 0} \sup_{\|E\|=1} \frac{|\lambda_i(X + tE) - \lambda_i(X)|}{t}.$$

- $\mathcal{O}_{ij}$ determines the eigenvalue correlation under the matrix Ornstein Uhlenbeck dynamics $X_t = e^{-t/2} X + \sqrt{1 - e^{-t}} \mathrm{Gin}$:

$$\mathrm{d}\lambda_i = \mathrm{d}\mathcal{M}_i - \frac{\lambda_i}{2} \mathrm{d}t, \qquad \mathrm{d}\langle \mathcal{M}_i, \overline{\mathcal{M}_j} \rangle_t = \mathcal{O}_{ij} \frac{\mathrm{d}t}{N},$$

*c.f.*, the Hermitian *Dyson Brownian motion*:

$$\mathrm{d}\lambda_i = \mathrm{d}M_i + \Big( -\frac{\lambda_i}{2} + \frac{1}{N} \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} \Big) \mathrm{d}t, \quad \mathrm{d}\langle M_i, \overline{M_j} \rangle_t = \delta_{ij} \frac{\mathrm{d}t}{N}.$$

# Previous results: Ginibre ensemble

- First moment for complex Ginibre [Chalker, Mehlig'98], [Bourgade,Dubach'20]:

$$\mathbf{E}(\mathcal{O}_{ii}|\lambda_i = z) \sim N(1-|z|^2),$$

$$\mathbf{E}(\mathcal{O}_{ij}|\lambda_i = z_1, \lambda_j = z_2) \sim -N\frac{1-z_1\overline{z_2}}{|\omega|^4}\frac{1-(1+|\omega|^2)e^{-|\omega|^2}}{1-e^{-|\omega|^2}},$$

with $\omega := N|z_1 - z_2|^2, \quad |z_1|, |z_2| < 1.$

Scales: microscopic $|z_1 - z_2| \approx N^{-1/2}$, mesoscopic $|z_1 - z_2| \gg N^{-1/2}$.

$$\mathbf{E}(\mathcal{O}_{ij}|\lambda_i = z_1, \lambda_j = z_2) \sim -\frac{1}{N}\frac{1-z_1\overline{z_2}}{|z_1-z_2|^4}. \qquad \text{[meso]}$$

- Condition on $\lambda_i = z$ inside the bulk $|z| < 1$ [Bourgade,Dubach'20]:

$$\frac{\mathcal{O}_{ii}}{N(1-|z|^2)} \to \frac{1}{\gamma_2} \sim \frac{e^{-\frac{1}{x}}}{x^3}\mathbb{1}_{x\geq 0}.$$

Real eigenvalues of real Ginibre with $1/\gamma_1$ [Fyodorov'18].

- A similar result for $\frac{\mathcal{O}_{ii}}{N^{1/2}}$ near the edge $|z| \approx 1$ [Fyodorov'18].

# Previous results: Ginibre ensemble

- Second moments of eigenvector overlaps [Bourgade,Dubach'20]:

$$\mathbf{E}(|\mathcal{O}_{ij}|^2|\lambda_i = z_1, \lambda_j = z_2) \sim \frac{N^2(1-|z_1|^2)(1-|z_2|^2)}{|\omega|^4},$$

$$\mathbf{E}(\mathcal{O}_{ii}\mathcal{O}_{jj}|\lambda_i = z_2, \lambda_j = z_2) \sim \frac{N^2(1-|z_1|^2)(1-|z_2|^2)}{|\omega|^4}\frac{1+|\omega|^4-e^{-|\omega|^2}}{1-e^{-|\omega|^2}}.$$

Quadradic decay on mesoscopic scales $|z_1 - z_2| \gg N^{-1/2}$:

$$\mathbf{E}(|\mathcal{O}_{ij}|^2|\lambda_i = z_1, \lambda_j = z_2) \sim \frac{(1-|z_1|^2)(1-|z_2|^2)}{|z_1-z_2|^4},$$

$$\mathbf{E}(\mathcal{O}_{ii}\mathcal{O}_{jj}|\lambda_i = z_1, \lambda_j = z_2) \sim \mathbf{E}(\mathcal{O}_{ii}|\lambda_i = z_1)\mathbf{E}(\mathcal{O}_{jj}|\lambda_j = z_2).$$

- Non-normal invariant ensemble [Benaych-Georges,Zeitouni'18]:

$$N|\lambda_i - \lambda_j|^2 \frac{|\langle \mathbf{r}_i, \mathbf{r}_j\rangle|^2}{\|\mathbf{r}_i\|^2\|\mathbf{r}_j\|^2} = \frac{Y}{\frac{Y}{N|\lambda_i-\lambda_j|^2}+1},$$

with $Y$ uniformly sub-Gaussian. If $|\lambda_i - \lambda_j| \gg N^{-1/2}$, then

$$\frac{\mathcal{O}_{ij}}{\sqrt{\mathcal{O}_{ii}\mathcal{O}_{jj}}} \approx \frac{|\langle \mathbf{r}_i, \mathbf{r}_j\rangle|^2}{\|\mathbf{r}_i\|^2\|\mathbf{r}_j\|^2} \approx \frac{Y}{N|\lambda_i-\lambda_j|^2} \ll 1.$$

# Previous results: beyond Ginibre ensemble

For general i.i.d. matrices without invariance property:

- Eigenvector delocalization [Rudelson,Vershynin'15],[Alt,Erdos,Kruger'18]...

$$\sup_{i=1}^{N} \frac{\|\mathbf{r}_i\|_\infty}{\|\mathbf{r}_i\|_2} \leq N^{-1/2+\epsilon}.$$

- Gaussian fluctuations for finite entries of eigenvector and asymptotic independent if $|\lambda_i - \lambda_j| \gg N^{-1/2}$ [Dubova,Yang,Yau,Yin'24],[Osman'24].

- Size of diagonal overlap [Erdos,Ji'24],[Cipolloni,Erdos,Henheik,Schroder'24]:

$$\mathbf{E}\mathcal{O}_{ii} \leq N^{1+\epsilon}, \qquad \mathcal{O}_{ii} \geq N^{1-\epsilon'}, \quad \text{w.h.p.}$$

- Distribution of diagonal overlap [Osman'24]:

$$\frac{\mathcal{O}_{ii}}{N(1-|z|^2)} \to \frac{1}{\gamma_\beta} \sim \frac{e^{-\frac{\beta}{x}}}{x^{\beta+1}} \mathbb{1}_{x \geq 0},$$

with $\beta = 1$ (real e.v.) and $\beta = 2$ (complex). A similar result for $|z| \approx 1$.

# Our results: off-diagonal overlaps for i.i.d. matrix

## Theorem (Cipolloni,Erdos,X.'24)

*Assume that $X$ is a real or complex i.i.d. matrix with $x_{ab} \overset{\mathrm{d}}{=} N^{-1/2}\chi$:*

$$\mathbf{E}\chi = 0, \quad \mathbf{E}|\chi|^2 = 1, \quad \mathbf{E}\big|\chi^p\big| \le C_p,$$

*additionally $\mathbf{E}\chi^2 = 0$ for complex. Then, with very high probability,*

$$\sup_{i,j\in[N]} \left(N|\lambda_i - \lambda_j|^2 + 1\right)\left[\frac{\big|\langle \mathbf{r}_i, \mathbf{r}_j\rangle\big|^2}{\|\mathbf{r}_i\|^2\|\mathbf{r}_j\|^2} + \frac{\big|\langle \mathbf{l}_i, \mathbf{l}_j\rangle\big|^2}{\|\mathbf{l}_i\|^2\|\mathbf{l}_j\|^2}\right] \le N^{\xi}.$$

*In particular, by a Cauchy-Schwarz inequality, this implies*

$$\sup_{i,j\in[n]} \left(N|\lambda_i - \lambda_j|^2 + 1\right)\frac{|\mathcal{O}_{ij}|}{\sqrt{\mathcal{O}_{ii}\mathcal{O}_{jj}}} \le N^{\xi}.$$

Corollary: there is a high prob event $\Xi$ s.t. for $|z_1 - z_2| \gg N^{-1/2}$,

$$\mathbf{E}\left(|\mathcal{O}_{ij}| \cdot \mathbf{1}_{\Xi}\Big|\lambda_i \approx z_1, \lambda_j \approx z_2\right) \le \begin{cases} \frac{N^{\xi}}{|z_1 - z_2|^2}, & \text{[complex]} \\ \frac{N^{\xi}}{|z_1 - z_2|^2}, & \text{[real, } \Im z_i > 0] \end{cases}$$

# Outline of the talk

1. Background and results
   - Hermitian v.s. non-Hermitian
   - Eigenvector overlaps: previous results
   - Eigenvector overlaps: new results

2. Sketch of proof
   - Hermitization: from eigenvector to singular vector
   - Local laws for multi-resolvents
   - Proof strategy: zig-zag

3. Summary

# Girko's Hermitization trick

- Girko's Hermitization trick [Girko'84]:

$$H^z := \begin{pmatrix} 0 & X-z \\ (X-z)^* & 0 \end{pmatrix} \in \mathbb{C}^{2N\times 2N}, \quad z \in \mathbb{C},$$

  with *chiral symmtric* eigenvalues and (normalized) eigenvectors:

$$\{\pm\sigma_i^z\}_{i=1}^N, \qquad \mathbf{w}_i^z = ((\mathbf{u}_i^z)^*, \pm(\mathbf{v}_i^z)^*)^* \in \mathbb{C}^{2N},$$

  where $\{\sigma_i^z\}$ are singular values of $X - z$ and $\{\mathbf{u}_i^z\}, \{\mathbf{v}_i^z\}$ are (normalized) left/right singular vectors in $\mathbb{C}^N$.

- Link non-Hermitian with Hermitian:

$$z \text{ is eigenvalue of } X \iff 0 \text{ is singular value of } X - z.$$

$$\text{eigenvector of } X \text{ for } \lambda_i = z \iff \text{singular vector of } X - z \text{ for } \sigma_1^z = 0.$$

- Reduce to study singular vector overlap of $X - z_1$ and $X - z_2$:

$$\sup_{i\in[N]} \frac{\langle \mathbf{l}_i, \mathbf{l}_j\rangle\langle \mathbf{r}_j, \mathbf{r}_i\rangle}{\|\mathbf{l}_i\|\|\mathbf{l}_j\|\|\mathbf{r}_i\|\|\mathbf{r}_j\|} = \sup_{z_1,z_2\in\mathrm{Spec}(X)} \langle \mathbf{u}_1^{z_1}, \mathbf{u}_1^{z_2}\rangle\langle \mathbf{v}_1^{z_1}, \mathbf{v}_1^{z_2}\rangle$$

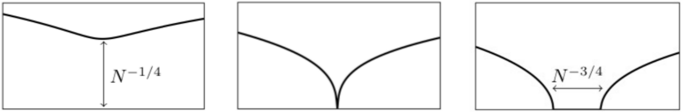
$$\lesssim \sup_{z_1,z_2\in\mathcal{D}} \{|\langle \mathbf{u}_1^{z_1}, \mathbf{u}_1^{z_2}\rangle|^2 + |\langle \mathbf{v}_1^{z_1}, \mathbf{v}_1^{z_2}\rangle|^2\}.$$

# Local law for resolvent of $H^z$

- Define the resolvent of $H^z$ by $G^z(w) := (H^z - w)^{-1}$, $w = E + i\eta$,

$$\frac{1}{N}\Im \mathrm{Tr} G^z(w) = \frac{1}{2N}\sum_i \frac{\eta}{(\sigma_i^z - E)^2 + \eta^2} = \frac{1}{2N}\sum_i \delta_\eta(\sigma_i^z - E).$$

$N \to \infty$ and $\eta \to 0$ gives the limiting spectral density of $H^z$, denoted by $\rho^z$.




$|z|<1$, $\eta_f = N^{-1}$ $|z|\approx 1$, $\eta_f = N^{-3/4}$ $|z|>1$, $\eta_f = N^{-2/3}$

- Local law for resolvent [Bourgade,Yau,Yin'14],[Alt,Erdos,Kruger'18]:

$$G^z(w) = M^z(w) + o(1), \qquad \eta_N \gg \eta_f,$$

where $M^z$ is the unique deterministic solution of *Matrix Dyson equation*, and $\eta_f = \eta_f(E) \sim (N\rho^z(E))^{-1}$ is the typical eigenvalue spacing at $E$.

# Observations from local law

- Eigenvector delocal. of $X$ = Singular vector delocal. of $H^z$ at zero:

  By a simple spectral decomposition of $H^z$ and choose $\eta \gg \eta_f$:

  $$\langle \mathbf{e}_k, \Im \mathrm{Tr} G^z(\mathrm{i}\eta)\mathbf{e}_k\rangle = \sum_i \frac{\eta}{(\sigma_i^z)^2+\eta^2}|\langle \mathbf{e}_k, \mathbf{w}_i^z\rangle|^2 \geq \frac{1}{\eta}|\langle \mathbf{e}_k, \mathbf{w}_1^z\rangle|^2,$$

  if $z \in \mathrm{Spec}(X)$, then $\sigma_1^z = 0$ and $\mathbf{w}_1^z = (\mathbf{u}_1^z, \mathbf{v}_1^z) = (\mathbf{l}_k, \mathbf{r}_k)$.

  Translate to non-Hermitian eigenvectors (normalized) [Alt,Erdos,Kruger'21]:

  $$\sup_k \{\|\mathbf{l}_k\|_\infty, \|\mathbf{r}_k\|_\infty\} = O(N^{-1/2}).$$

- Eigenvector overlap of $X$ = singular vector overlap $H^z$ at zero:

  To study singular vector overlap of $X - z_i$ near zero, we need

  $$G^{z_1}(\mathrm{i}\eta)G^{z_2}(\mathrm{i}\eta) \not\approx M^{z_1}M^{z_2}$$

# Reduce to multi-resolvent bound

- By spectral decomposition of $G^{z_i}$ and eigenvalue rigidity $\sigma_1^{z_i} \lesssim \eta_f$:

$$\frac{1}{N}\mathrm{Tr}\Big[\Im G^{z_1}(\mathrm{i}\eta_1)\Im G^{z_2}(\mathrm{i}\eta_2)\Big] = \frac{4}{N}\sum_{j,k=1}^{N} \frac{\eta_1\eta_2\big(|\langle \mathbf{u}_j^{z_1}, \mathbf{u}_k^{z_2}\rangle|^2 + |\langle \mathbf{v}_j^{z_1}, \mathbf{v}_k^{z_2}\rangle|^2\big)}{\big((\sigma_j^{z_1})^2+\eta_1^2\big)\big((\sigma_k^{z_2})^2+\eta_2^2\big)}$$

$$\gtrsim \frac{1}{N\eta_1\eta_2}\big[|\langle \mathbf{u}_1^{z_1}, \mathbf{u}_1^{z_2}\rangle|^2 + |\langle \mathbf{v}_1^{z_1}, \mathbf{v}_1^{z_2}\rangle|^2\big].$$

- We hence conclude that

$$\sup_{|z_i|\le 1}(N|z_1-z_2|^2+1)\big[|\langle \mathbf{u}_1^{z_1}, \mathbf{u}_1^{z_2}\rangle|^2 + |\langle \mathbf{v}_1^{z_1}, \mathbf{v}_1^{z_2}\rangle|^2\big]$$

$$\lesssim \sup_{|z_i|\le 1}(N|z_1-z_2|^2+1)(N\eta_1\eta_2)\frac{1}{N}\mathrm{Tr}\Big[\Im G^{z_1}(\mathrm{i}\eta_1)\Im G^{z_2}(\mathrm{i}\eta_2)\Big]$$

$$\lesssim \sup_{|z_i|\le 1}\big(|z_1-z_2|^2+N^{-1}\big)\frac{N\eta_1\rho_1 N\eta_2\rho_2}{|z_1-z_2|^2} \lesssim 1.$$

- No cheap way to use resolvent identity reducing to one-resolvent.

# Multi–resolvent local laws

## Theorem (Cipolloni,Erdos,X'24)

*For any deterministic bounded matrices and vectors, the following*

$$\left|\langle \mathbf{x}, (G^{z_1}(\mathrm{i}\eta_1)A_1G^{z_2}(\mathrm{i}\eta_2) - M_{12}^{A_1})\mathbf{y}\rangle\right| \prec \frac{1}{\sqrt{N\eta}}\frac{1}{\sqrt{\gamma}},$$

$$\left|\frac{1}{N}\mathrm{Tr}\Big[(\Im G^{z_1}(\mathrm{i}\eta_1)A_1\Im G^{z_2}(\mathrm{i}\eta_2) - \widehat{M}_{12}^{A_1})A_2\Big]\right| \prec \frac{1}{\sqrt{N\ell}}\frac{\rho_1\rho_2}{\widehat{\gamma}},$$

*hold uniformly for* $|z_i| \le 1 + N^{-1/2+\tau}$ *and* $\ell = \min_{i=1,2}\rho_i|\eta_i| \ge N^{-1+\epsilon}$, *with*

$$\|M_{12}^{A_1}\| \lesssim \frac{1}{\gamma}, \qquad \gamma := \frac{|z_1-z_2|^2 + \rho_1|\eta_1| + \rho_2|\eta_2|}{|z_1-z_2| + \rho_1^2 + \rho_2^2},$$

$$\|\widehat{M}_{12}^{A_1}\| \lesssim \frac{\rho_1\rho_2}{\widehat{\gamma}}, \qquad \widehat{\gamma} := |z_1-z_2|^2 + \rho_1|\eta_1| + \rho_2|\eta_2|.$$

Compare to standard local law without $|z_1 - z_2|$ decay:

$$\left|\langle \mathbf{x}, (G^{z_1}(\mathrm{i}\eta_1)A_1G^{z_2}(\mathrm{i}\eta_2) - M_{12}^{A_1})\mathbf{y}\rangle\right| \prec \frac{1}{\sqrt{N\eta}}\sqrt{\frac{\rho}{\eta}}.$$

Note $1/\sqrt{\gamma}$ is better in the mesoscopic scale $|z_1 - z_2| \gg N^{-1/2}$.

# Proof: Zig-Zag strategy

- Global law for multi-resolvent $\eta_i \sim 1$ (easy to check):

$$\left|\left\langle \mathbf{x}, (G^{z_1}(\mathrm{i}\eta_1)A_1G^{z_2}(\mathrm{i}\eta_2) - M_{12}^{A_1})\mathbf{y}\right\rangle\right| \prec \frac{1}{\sqrt{N}}.$$

- Zig step: given the random OU-matrix flow:

$$\mathrm{d}X_t = -\frac{1}{2}X_t\mathrm{d}t + \frac{\mathrm{d}B_t}{\sqrt{N}}, \qquad X_0 = X, \qquad W_t := \begin{pmatrix} 0 & X_t \\ X_t^* & 0 \end{pmatrix},$$

  find a proper deterministic *characteristic flow* to reduce $\eta_i$ to local scales

$$\partial_t\Lambda_t = -\mathcal{S}[M(\Lambda_t)] - \frac{\Lambda_t}{2}, \qquad \Lambda_t := \begin{pmatrix} \mathrm{i}\eta_t & z_t \\ \overline{z_t} & \mathrm{i}\eta_t \end{pmatrix},$$

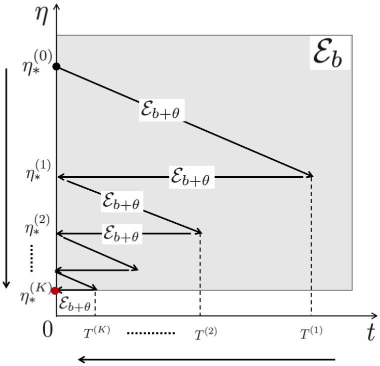
  such that desired error bound of $G_{1,t}G_{2,t}$ is kept along the flow.

- Zag step: remove added Gaussian part (worse for large $t$ and small $\eta$).

$$X_t \stackrel{\mathrm{d}}{=} e^{-\frac{t}{2}}X + \sqrt{1-e^{-t}}\mathrm{Gin}(\mathbb{C}).$$

# Zig-Zag strategy

- Zig and Zag fight agaist each other, so we run zig-zag iteratively to reduce global scale $\eta \sim 1$ to optimal local scales $\eta \sim N^{-1}$.




But it is still not enough to gain full $|z_1 - z_2|$ quadratic decay!

# Bootstrap Zig-Zag process

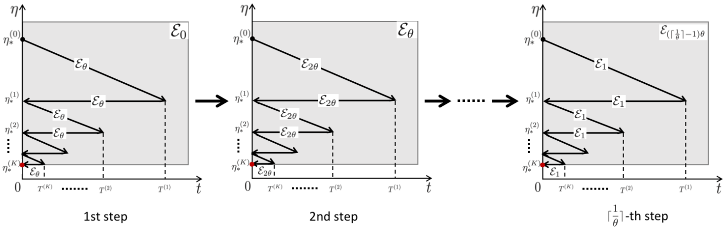
## Proposition

*Assuming* $\gamma \gtrsim \eta_*/\rho^*$ *and the following hold for some* $0 \le b < 1$

$$\left|\left\langle \mathbf{x}, (G^{z_1}(\mathrm{i}\eta_1)A_1G^{z_2}(\mathrm{i}\eta_2) - M_{12}^{A_1})\mathbf{y}\right\rangle\right| \prec \frac{(\rho^*)^{\frac{1-b}{2}}}{\sqrt{n}(\eta_*)^{\frac{3-b}{2}}\gamma^{\frac{b}{2}}},$$

*uniformly in* $\min_{i=1}^{2}\{|\eta_i|\rho_i\} \ge n^{-1+\epsilon}$, *then the same also hold for a larger* $b' = b + \theta$ *with* $0 < \theta < \epsilon/10$.

Start with the standard local law without $|z_1 - z_2|$-decorrelation ($b = 0$): Iterate zig-zag for $O(\epsilon^{-1})$ times to increase $b = 0$ to $b = 1$.



# Outline of the talk

# Summary

**Results:** Define $\mathcal{O}_{ij} := \langle \mathbf{r}_j, \mathbf{r}_i \rangle \langle \mathbf{l}_j, \mathbf{l}_i \rangle$ with $\langle \mathbf{l}_i, \mathbf{r}_j \rangle = \delta_{ij}$.

1) Extend [Bourgade,Dubach'20] for Ginibre to iid for $|z_1 - z_2| \gg N^{-1/2}$:

$$\mathbf{E}\Big( |\mathcal{O}_{ij}| \Big| \lambda_i \approx z_2, \lambda_j \approx z_2 \Big) = O\Big( \frac{1}{|z_1 - z_2|^2} \Big).$$

2) Extend Ginibre result [Benaych-Georges, Zeitouni'18] to iid cases:

$$\frac{\sqrt{N} |\lambda_i - \lambda_j| |\langle \mathbf{r}_i, \mathbf{r}_j \rangle|}{\|\mathbf{r}_i\| \|\mathbf{r}_j\|} = O(1).$$

**Proof:** use Hermitization trick to reduce eigenvectors to singular vectors:

a) further reduce to study $\Im G^{z_1}(\mathrm{i}\eta_1) \Im G^{z_2}(\mathrm{i}\eta_2)$ using eigenvalue rigidity;

b) use zig-zag iteratively to derive the local laws for multi-resolvents.


Happy Birthday Peter!