KU Leuven (Belgium), Department of Mathematics

Multiple orthogonal polynomial ensembles of derivative type

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Motivation

Main result: **characterization** of multiple orthogonal polynomial ensembles of derivative type

Motivation:

- many important random matrix models arise in this way
  E.g. GUE, LUE, JUE, their sums and products
- obtain new models with good properties
- opens up the road to develop new notions of derivative type

Overview

Overview:

- Polynomial ensembles (of derivative type)
- A finite free probability perspective
- Characterization
- Discrete notions of derivative type

PE: some history

Question: can we describe the SSV of a **product** of two random matrices X_1 and X_2?

- Akemann–Kieburg–Wei (2013): $X_1, X_2 =$ Ginibre matrix
- Kuijlaars–Stivigny (2014): SSV$(X_1) \sim$ PE, $X_2 =$ Ginibre matrix
- Kieburg–Kuijlaars–Stivigny (2015): SSV$(X_1) \sim$ PE, $X_2 =$ truncated random unitary matrix
- ! Kieburg–Kösters (2016): SSV$(X_1) \sim$ PE, SSV$(X_2) \sim \mathrm{PE}_{\mathrm{MDT}}$

Analogue for the EV of sums of random matrices by Kuijlaars–Román (2016)

PE: definition

Polynomial ensemble [PE($w_1, \dots, w_n$)]: probability density on $\mathbb{R}^n$ of the form

$$\mathcal{P}(\vec{x}) = \frac{1}{Z_n} \Delta_n(\vec{x}) \det[w_j(x_k)]_{j,k=1}^{n} \geq 0, \quad \vec{x} \in \mathbb{R}^n.$$

$\longrightarrow$ **biorthogonal ensemble** (determinantal point process) with correlation kernel

$$K_n(x, y) = \sum_{k=0}^{n-1} P_k(x) Q_k(y),$$

in terms of monic $P_k \in \mathbb{R}[x]$ with $\deg P_k = k$ and $Q_l \in \operatorname{span}\{w_j\}_{j=1}^{l+1}$ that satisfy

$$\int_0^\infty P_k(x) Q_l(x) dx = \delta_{k,l}, \quad k, l \in \{0, \dots, n-1\}.$$

Important property: if $\mathrm{EV}(X) \sim \mathrm{PE}(w_1, \dots, w_n)$, then

$$P_n(x) = \mathbb{E}[\det(x I_n - X)].$$

PE: special cases

Polynomial ensemble [PE($w_1, \ldots, w_n$)]: probability density on $\mathbb{R}^n$ of the form

$$\mathcal{P}(\vec{x}) = \frac{1}{Z_n} \Delta_n(\vec{x}) \det[w_j(x_k)]_{j,k=1}^n \geq 0, \quad \vec{x} \in \mathbb{R}^n.$$

Special cases:

- OPE(w) (Coulomb gas with $\beta = 2$): $x^{k-1} w(x)$ for $k = 1, \ldots, n$
- MOPE($w_1, \ldots, w_r$): $x^{k-1} w_j(x)$ for $k = 1, \ldots, n_j$ and $j = 1, \ldots, r$ with $\vec{n} \in \mathcal{S}^r$ s.t. $|\vec{n}| = n$
- $\text{PE}_{\text{DT}}(\omega)$: $D^{(k-1)}\omega$ for $k = 1, \ldots, n$

$\mathrm{PE}_{\mathrm{DT}}$: definition

Polynomial ensemble of derivative type $[\mathrm{PE}_{\mathrm{DT}}(\omega)]$: probability density on Λ^n of the form

$$\mathcal{P}(\vec{x}) = \frac{1}{Z_n}\Delta_n(\vec{x})\det[(D^{j-1}\omega)(x_k)]_{j,k=1}^n \geq 0, \quad \vec{x} \in \Lambda^n,$$

for a certain differential operator D on functions supported on $\Lambda \subset \mathbb{R}$.

Main notions:

- $\mathrm{PE}_{\mathrm{MDT}}(\omega)$: $(Df)(x) = -xf'(x)$ on $\Lambda = (0, \infty)$
  $\longrightarrow$ SSV of (products of) invertible complex random matrices
- $\mathrm{PE}_{\mathrm{ADT}}(\omega)$: $(Df) = f'(x)$ on $\Lambda = \mathbb{R}$
  $\longrightarrow$ EV of (sums of) Hermitian random matrices

Other notions:

- Förster–Kieburg–Kösters (2017): SSV of (products of) complex rectangular random matrices
- Kieburg–Li–Zhang–Forrester (2020): SSV (of products) of random unitary matrices

PE_{MDT}: examples

Polynomial ensemble of multiplicative derivative type [$\text{PE}_{\text{MDT}}(\omega)$]:

$$\mathcal{P}(\vec{x}) = \frac{1}{Z_n} \Delta_n(\vec{x}) \det[(-x_k \frac{d}{dx_k})^{j-1} \omega(x_k)]_{j,k=1}^n \geq 0, \quad \vec{x} \in (0, \infty)^n.$$

Examples:

- EV of **LUE** has $\omega(x) = x^a e^{-x}$ on $(0, \infty)$
  $\longrightarrow$ SSV of Ginibre matrix
- EV of **JUE** has $\omega(x) = x^a (1-x)^{n+b}$ on $(0, 1)$
  $\longrightarrow$ SSV of truncated random unitary matrix

Characterization by Förster–Kieburg–Kösters (2017): TFAE,

i) ω gives rise to an n-point PE_{MDT}

ii) ω is a multiplicative Pólya frequency function of order n

$\mathrm{PE}_{\mathrm{MDT}}$: properties

Polynomial ensemble of multiplicative derivative type [$\mathrm{PE}_{\mathrm{MDT}}(\omega)$]:

$$\mathcal{P}(\vec{x}) = \frac{1}{Z_n}\Delta_n(\vec{x})\,\det[(-x_k\frac{d}{dx_k})^{j-1}\omega(x_k)]_{j,k=1}^{n} \geq 0, \quad \vec{x} \in (0,\infty)^n.$$

(De)composition properties:

- If $\mathrm{SSV}(X_1) \sim \mathrm{PE}_{\mathrm{MDT}}(\omega_1)$ and $\mathrm{SSV}(X_2) \sim \mathrm{PE}_{\mathrm{MDT}}(\omega_2)$, then

$$\mathrm{SSV}(X_1X_2) \sim \mathrm{PE}_{\mathrm{MDT}}(\omega_1 *_{\mathcal{M}} \omega_2).$$

- If $\mathrm{SSV}(X_1) \sim \mathrm{PE}(w_1,\ldots,w_n)$ and $\mathrm{SSV}(X_2) \sim \mathrm{PE}_{\mathrm{MDT}}(\omega)$, then

$$\mathrm{SSV}(X_1X_2) \sim \mathrm{PE}(w_1 *_{\mathcal{M}} \omega,\ldots,w_n *_{\mathcal{M}} \omega).$$

PE_{MDT}: properties

Polynomial ensemble of multiplicative derivative type:

$$\mathcal{P}(\vec{x}) = \frac{1}{Z_n} \Delta_n(\vec{x}) \det[(-x\frac{d}{dx_k})^{j-1} \omega(x_k)]_{j,k=1}^n \geq 0, \quad \vec{x} \in (0,\infty)^n.$$

Biorthogonal system:

- If $\text{SSV}(X) \sim \text{PE}_{\text{MDT}}(\omega)$, then

$$P_j^X(x) = \sum_{k=0}^{j} (-1)^{j-k} \binom{j}{k} \frac{(\mathcal{M}\omega_n)(j+1)}{(\mathcal{M}\omega_n)(k+1)} x^k, \quad Q_j^X(x) = \frac{1}{j!\mathcal{M}\omega_n(j+1)} (-x\frac{d}{dx})^j \omega(x).$$

$\longrightarrow$ double integral representation for kernel

- If $\text{SSV}(X_1) \sim \text{PE}(w_1, \ldots, w_n)$ and $\text{SSV}(X_2) \sim \text{PE}_{\text{MDT}}(\omega)$, then

$$P_j^{X_1 X_2} = P_j^{X_1} \boxtimes_j P_j^{X_2}, \quad Q_j^{X_1 X_2} = Q_j^{X_1} *_{\mathcal{M}} \omega.$$

$\longrightarrow$ connection to finite free probability

Products of random matrices & finite free probability

Finite free multiplicative convolution:

$$(p_n^1 \boxtimes_n p_n^2)(x) = \sum_{k=0}^{n} \frac{p_n^1[k] p_n^2[k]}{(-1)^{n-k}\binom{n}{k}} x^k, \qquad \text{for } p_n^j(x) = \sum_{k=0}^{n} p_n^j[k] x^k.$$

Marcus–Spielman–Srivastava (2022): for $n \times n$ normal matrices X_1 and X_2, we have

$$\det(xI_n - X_1) \boxtimes_n \det(xI_n - X_2) = \mathbb{E}_{Q \in U(n)}[\det(xI_n - X_1 Q X_2 Q^*)].$$

Thus, for $n \times n$ independent Hermitian unitarily invariant random matrices X_1 and X_2:

$$\mathbb{E}[\det(xI_n - X_1)] \boxtimes_n \mathbb{E}[\det(xI_n - X_2)] = \mathbb{E}[\det(xI_n - X_1 X_2)].$$

$\longrightarrow$ finite version of **free multiplicative convolution law**

PE_{ADT}: examples

Polynomial ensemble of additive derivative type [$\text{PE}_{\text{ADT}}(\omega)$]:

$$\mathcal{P}(\vec{x}) = \frac{1}{Z_n}\Delta_n(\vec{x})\,\det[\omega^{(j-1)}(x_k)]_{j,k=1}^{n} \geq 0, \quad \vec{x} \in \mathbb{R}^n.$$

Examples:

- EV of **GUE** has $\omega(x) = e^{-x^2}$ on $\mathbb{R}$
- EV of **LUE** has $\omega(x) = x^{n+a}e^{-x}$ on $(0,\infty)$

Characterization by Förster–Kieburg–Kösters (2017):

i) ω gives rise to an n-point PE_{ADT}

ii) ω is an additive Pólya frequency function of order n

$\mathrm{PE_{ADT}}$: properties

Polynomial ensemble of additive derivative type [$\mathrm{PE_{ADT}}(\omega)$]:

$$\mathcal{P}(\vec{x}) = \frac{1}{Z_n}\Delta_n(\vec{x})\,\det[\omega^{(j-1)}(x_k)]_{j,k=1}^{n} \geq 0, \quad \vec{x} \in \mathbb{R}^n.$$

(De)composition properties:

- If $\mathrm{EV}(X_1) \sim \mathrm{PE_{ADT}}(\omega_1)$ and $\mathrm{EV}(X_2) \sim \mathrm{PE_{ADT}}(\omega_2)$, then

$$\mathrm{EV}(X_1 + X_2) \sim \mathrm{PE_{ADT}}(\omega_1 *_{\mathcal{L}} \omega_2).$$

- If $\mathrm{EV}(X_1) \sim \mathrm{PE}(w_1, \ldots, w_n)$ and $\mathrm{EV}(X_2) \sim \mathrm{PE_{ADT}}(\omega)$, then

$$\mathrm{EV}(X_1 + X_2) \sim \mathrm{PE}(w_1 *_{\mathcal{L}} \omega, \ldots, w_n *_{\mathcal{L}} \omega).$$

$\mathrm{PE}_{\mathrm{ADT}}$: correlation kernel

Polynomial ensemble of additive derivative type:

$$\mathcal{P}(\vec{x}) = \frac{1}{Z_n} \Delta_n(\vec{x}) \det[\omega^{(j-1)}(x_k)]_{j,k=1}^{n} \geq 0, \quad \vec{x} \in \mathbb{R}^n.$$

Biorthogonal system:

- If $\mathrm{EV}(X) \sim \mathrm{PE}_{\mathrm{ADT}}(\omega)$, then

$$P_j^X(x) = \left(x - \frac{d}{dt}\right)^j \left[\frac{1}{\mathcal{L}\omega(t)}\right]_{t=0}, \quad Q_j^X(x) = \frac{(-1)^j}{j!}\omega^{(j)}(x).$$

$\longrightarrow$ double integral representation for kernel

- If $\mathrm{EV}(X_1) \sim \mathrm{PE}(w_1, \ldots, w_n)$ and $\mathrm{EV}(X_2) \sim \mathrm{PE}_{\mathrm{ADT}}(\omega)$, then

$$P_j^{X_1+X_2} = P_j^{X_1} \boxplus_j P_j^{X_2}, \quad Q_j^{X_1+X_2} = Q_j^{X_1} *_{\mathcal{L}} \omega.$$

$\longrightarrow$ connection to finite free probability

Sums of random matrices & finite free probability

Finite free additive convolution:

$$(p \boxplus_n q)(x) = \frac{1}{n!} \sum_{k=0}^{n} p^{(k)}(x) q^{(n-k)}(0).$$

Marcus–Spielman–Srivastava (2022): for $n \times n$ normal matrices X_1 and X_2, we have

$$\det(xI_n - X_1) \boxplus_n \det(xI_n - X_2) = \mathbb{E}_{Q \in U(n)}[\det(xI_n - (X_1 + QX_2Q^*))].$$

Thus, for $n \times n$ independent Hermitian unitarily invariant random matrices X_1 and X_2:

$$\mathbb{E}[\det(xI_n - X_1)] \boxplus_n \mathbb{E}[\det(xI_n - X_2)] = \mathbb{E}[\det(xI_n - (X_1 + X_2))].$$

$\longrightarrow$ finite version of **free additive convolution law**

MOPE_{MDT}: characterization

Theorem

If $w_1, \ldots, w_r \in L^1_{\mathcal{M},\Sigma}(\mathbb{R}_{>0})$ give rise to an n-point MOPE for all $n \in \mathbb{N}$, then TFAE:

i) for all $n \in \mathbb{N}$, the n-point ensemble is of multiplicative derivative type,

ii) the **Mellin transforms** *of $w_1, \ldots, w_r$ are given by*

$$\mathcal{M}w_j(s) = c^s \prod_{i=1}^{r} \frac{\Gamma(s+a_i)^{d_1(i)}}{\Gamma(s+b_i)^{d_2(i)}} \frac{s^{j-1}}{\prod_{i=1}^{j}(s+b_i)^{d_2(i)}}, \quad s \in \Sigma,$$

with $d_1, d_2 : \{1, \ldots, r\} \to \{0, 1\}$ with $d_1 = 1$ or $d_2 = 1$ on $\{1, \ldots, r\}$.

MOPE_{MDT}: corollaries

An any-point $\text{MOPE}_{\text{MDT}}(w_1, \ldots, w_r)$ has

$$\mathcal{M}w_j(s) = c^s \prod_{i=1}^{r} \frac{\Gamma(s+a_i)^{d_1(i)}}{\Gamma(s+b_i)^{d_2(i)}} \frac{s^{j-1}}{\prod_{i=1}^{j}(s+b_i)^{d_2(i)}}, \quad s \in \Sigma.$$

Corollaries:

- the only (any-point) OPE_{MDT} are the LUE and JUE (up to a linear transformation)
- most MOPE_{MDT} decompose as **products of LUE and JUE**
  $\Rightarrow$ Meijer G-ensemble (as $n \to \infty$)

MOPE_{ADT}: characterization

Theorem

If $w_1, \ldots, w_r \in L^1_{\mathcal{L},\Sigma}(\mathbb{R})$ give rise to an n-point MOPE for all $n \in \mathbb{N}$, then TFAE:

i) for all $n \in \mathbb{N}$, the n-point ensemble is of additive derivative type,

ii) the **Laplace transforms** *of $w_1, \ldots, w_r$ are given by*

$$\mathcal{L}w_j(s) = \exp\left(c \int_{s_0}^{s} \prod_{i=1}^{r} \frac{(t+a_i)^{d_1(i)}}{(t+b_i)^{d_2(i)}}\, dt\right) \frac{s^{j-1}}{\prod_{i=1}^{j}(s+b_i)^{d_2(i)}}, \quad s \in \Sigma,$$

with $d_1, d_2 : \{1, \ldots, r\} \to \{0, 1\}$ with $d_1 = 1$ or $d_2 = 1$ on $\{1, \ldots, r\}$.

$\mathrm{MOPE}_{\mathrm{ADT}}$: corollaries

An any-point $\mathrm{MOPE}_{\mathrm{ADT}}(w_1, \dots, w_r)$ has

$$\mathcal{L}w_j(s) = \exp\left(c \int_{s_0}^{s} \prod_{i=1}^{r} \frac{(t+a_i)^{d_1(i)}}{(t+b_i)^{d_2(i)}} dt\right) \frac{s^{j-1}}{\prod_{i=1}^{j}(s+b_i)^{d_2(i)}}, \quad s \in \Sigma.$$

Corollaries:

- the only (any-point) $\mathrm{OPE}_{\mathrm{ADT}}$ are the LUE and GUE (up to an affine transformation)
- most $\mathrm{MOPE}_{\mathrm{ADT}}$ decompose as **sums of basic $\mathbf{MOPE}_{\mathbf{ADT}}$** w.r.t.
  - Airy-like functions ($p \geq 2$)

$$\omega(x) = \int_{\mathcal{C}} e^{s^p + sx} \frac{ds}{2\pi i}, \quad x \in \mathbb{R},$$

  - I-Bessel-like functions ($q \geq 0$)

$$\omega(x) = e^{-x} \sum_{k=0}^{\infty} \frac{x^{qk+c}}{k!\Gamma(qk+c+1)}, \quad x \in \mathbb{R}_{>0}.$$

$\longrightarrow$ many new examples

MOPE_{MDT} & MOPE_{ADT}: set-up

Set-up: consider sets of functions $\{v_j\}_{j=1}^n$ of **derivative type**

- MDT: if there exists $\omega \in C^{n-2}(\mathbb{R}_{>0})$, with $\omega^{(n-2)} \in AC_{\text{loc}}(\mathbb{R}_{>0})$, s.t.

$$\operatorname{span}\{v_j(x)\}_{j=1}^n = \operatorname{span}\{(x\frac{d}{dx})^{j-1}\omega(x)\}_{j=1}^n, \quad \text{a.e. } x \in \mathbb{R}_{>0}.$$

- ADT: if there exists $\omega \in C^{n-2}(\mathbb{R})$, with $\omega^{(n-2)} \in AC_{\text{loc}}(\mathbb{R})$, s.t.

$$\operatorname{span}\{v_j(x)\}_{j=1}^n = \operatorname{span}\{\omega^{(j-1)}(x)\}_{j=1}^n, \quad \text{a.e. } x \in \mathbb{R}.$$

$\longrightarrow$ specify later to sets of the form

$$\mathcal{W}_{\vec{n}}(\vec{w}) = \operatorname{span}\{\mathbb{R} \to \mathbb{R} : x \mapsto x^{k-1}w_j(x) \mid k = 1, \ldots, n_j, j = 1, \ldots, r\}.$$

MOPE_{MDT} & MOPE_{ADT}: idea

Idea:

1. convert analytic conditions into algebraic ones by applying the **appropriate transform**
2. uncover hidden derivative type structure
3. use the additional structure of $\mathcal{W}_{\vec{n}}(\vec{w})$ to characterize $\vec{w}$

Mellin transform

Idea:

1. convert analytic conditions into algebraic ones by applying the appropriate transform

Mellin transform:

$$\mathcal{M}f(s) = \int_0^\infty f(x)x^{s-1}dx, \quad s \in \Sigma.$$

Standard properties:

- (convolution) there is a convolution $f *_{\mathcal{M}} g$ s.t.

$$\mathcal{M}(f *_{\mathcal{M}} g)(s) = \mathcal{M}f(s)\mathcal{M}g(s).$$

- (inversion) there is an inversion formula
- (**differentiation**) $\mathcal{M}Df(s) = s\mathcal{M}f(s)$ for $(Df)(x) = -xf'(x)$

MDT in the Mellin space

Idea:

1. convert analytic conditions into **algebraic** ones by applying the appropriate transform

Proposition

If $v_1, \ldots, v_n \in L^1_{\mathcal{M},\Sigma}(\mathbb{R}_{>0})$, then TFAE:

i) $\{v_j\}_{j=1}^n$ *is of multiplicative derivative type w.r.t. some* ω,

ii) *there exists* $\hat{\omega} : \Sigma \to \mathbb{C}$ *s.t.*

$$\mathit{span}\{\mathcal{M}v_j(s)\}_{j=1}^n = \mathit{span}\{s^{j-1}\hat{\omega}(s)\}_{j=1}^n, \quad s \in \Sigma.$$

In that case, $\mathcal{M}\omega = \hat{\omega}$ *on* Σ.

Laplace transform

Idea:

1. convert analytic conditions into algebraic ones by applying the appropriate transform

Laplace transform:

$$\mathcal{L}f(s) = \int_{-\infty}^{\infty} f(x)e^{-sx}dx, \quad s \in \Sigma.$$

Standard properties:

- (convolution) there is a convolution $f *_{\mathcal{L}} g$ such that

$$\mathcal{L}(f *_{\mathcal{L}} g)(s) = \mathcal{L}f(s)\mathcal{L}g(s).$$

- (inversion) there is an inversion formula
- (**differentiation**) $\mathcal{L}Df(s) = s\mathcal{L}f(s)$ for $(Df)(x) = f'(x)$

ADT in the Laplace space

Idea:

1. convert analytic conditions into **algebraic** ones by applying the appropriate transform

Proposition

If $v_1, \dots, v_n \in L^1_{\mathcal{L},\Sigma}(\mathbb{R})$, then TFAE:

i) $\{v_j\}_{j=1}^n$ *is of additive derivative type w.r.t. some* ω,

ii) *there exists* $\hat{\omega} : \Sigma \to \mathbb{C}$ *s.t.*

$$span\{\mathcal{L}v_j(s)\}_{j=1}^n = span\{s^{j-1}\hat{\omega}(s)\}_{j=1}^n, \quad s \in \Sigma.$$

In that case, $\mathcal{L}\omega = \hat{\omega}$ *on* Σ.

Sets of functions of MDT & ADT

Idea:

2. uncover hidden **derivative type structure**

Proposition

If, for all $n \in \{1, \ldots, N\}$, there exists $\hat{\omega}_n : \Sigma \to \mathbb{R}$ s.t.

$$span\{\mathcal{T} v_j(s)\}_{j=1}^{n} = span\{s^{k-1}\hat{\omega}_n(s)\}_{k=1}^{n}, \quad s \in \Sigma,$$

then there exists $d : \{1, \ldots, N-1\} \to \{0, 1\}$ and $b_i \in \mathbb{R}$ s.t., for all $n \in \{1, \ldots, N\}$,

$$\hat{\omega}_n(s) = \frac{b_0 \mathcal{T} v_1(s)}{\prod_{i=1}^{n-1}(s + b_i)^{d(i)}}, \quad s \in \Sigma.$$

Interpretation: up to some linear combinations of the weights, we have

$$\frac{\mathcal{T}\tilde{v}_{n+1}(s)}{\mathcal{T}\tilde{v}_n(s)} = \begin{cases} s, & d(n) = 0, \\ \frac{1}{s+b_n}, & d(n) = 1, \end{cases} \qquad s \in \Sigma.$$

Sets $\mathcal{W}_{\vec{n}}(\vec{w})$ of MDT & ADT

Idea:

3. use the additional structure of $\mathcal{W}_{\vec{n}}(\vec{w})$ to identify $\vec{w}$

If additionally, for all $\vec{n} \in \mathcal{S}^r$ with $|\vec{n}| = n$,

$$\operatorname{span}\{v_j(x)\}_{j=1}^n = \operatorname{span}\{x^{k-1} w_j(x) \mid k = 1, \ldots, n_j, j = 1, \ldots, r\},$$

then

- MDT:

$$\mathcal{M}w_1(s+1) = c \prod_{i=1}^r \frac{(t+a_i)^{d_1(i)}}{(t+b_i)^{d_2(i)}} \mathcal{M}w_1(s), \quad s \in \Sigma,$$

- ADT:

$$(\mathcal{L}w_1)'(s) = c \prod_{i=1}^r \frac{(t+a_i)^{d_1(i)}}{(t+b_i)^{d_2(i)}} \mathcal{L}w_1(s), \quad s \in \Sigma.$$

$\longrightarrow$ allows us to **characterize** $\vec{w}$

Collaborative project: overview

First meeting: tomorrow at 15h30 (room 139 of MATRIX House)

Goal: develop **discrete notions of derivative type**

Motivation: several non-intersecting path models seem to fit into this framework
$\longrightarrow$ deepen understanding of these models (! tiling models)

Subgoals:

1. fit existing non-intersecting path models in this framework
2. study associated kernels
   ? double integral representation of kernel $\longrightarrow$ asymptotic analysis
3. describe associated (de)composition properties
   ? hierarchy
4. describe implications for the initial models

Collaborative project: discrete notions of DT & MOP

Goal: develop discrete notions of derivative type

1. fit existing non-intersecting path models in this framework $\longrightarrow$ MOP

Examples of interest:

- Johansson (2001): uniform hexagon tilings $\rightarrow$ Hahn
- Duits–Duse–Liu (2024): non-uniform, non-periodic hexagon tilings $\rightarrow$ q-Racah
- Duits–Fahs–Kozhan (2021): random growth models $\rightarrow$ descendants of multiple Hahn

$\longrightarrow$ $\mathrm{MOPE}_{\mathrm{DT}}$?

Question: what about the other polynomials in the (multiple) (q-)**Askey scheme**?

Collaborative project: discrete notions of DT & MOP

Example: **Hahn ensemble**, which is the discrete PE on $\{0, \dots, N\}$ w.r.t.

$$v_j(x) = (-x)_j \frac{\Gamma(x+\alpha+1)}{\Gamma(x+1)} \frac{\Gamma(N-x+\beta+1)}{\Gamma(N-x+1)}.$$

Consider a shifted discrete analogue of the Mellin transform

$$\mathcal{M}_N^\alpha f(s) = \sum_{x=0}^{N} f(x) \frac{\Gamma(x+s)}{\Gamma(x+\alpha+1)}.$$

Branquinho–Díaz–Foulquié-Moreno–Mañas–W. (2025): for some (explicit) $\hat{\omega}_{N,n}$, we have

$$\operatorname{span}\{\mathcal{M}_N^\alpha v_j(s)\}_{j=1}^n = \operatorname{span}\{s^{k-1}\hat{\omega}_{N,n}(s)\}_{k=1}^n, \quad \operatorname{Re}(s) > 0.$$

$\longrightarrow$ suggests underlying derivative type structure in terms of $\mathcal{T} = \mathcal{M}_N^\alpha$

Question: what is the associated differential operator?

Collaborative project: discrete notions of DT & free probability

Goal: develop discrete notions of derivative type

3. describe associated (de)composition properties $\longrightarrow$ finite free probability

Observation: $\boxplus_n$ & $\boxtimes_n$ can be defined using the differential operator for ADT & MDT

- Mirabelli (2020):

$$(p_n^1 \boxplus_n p_n^2)(x) = \hat{p}_n^1(\frac{d}{dx})\hat{p}_n^2(\frac{d}{dx})x^n, \quad \text{for } p_n^j(x) = \hat{p}_n^j(-x\frac{d}{dx})x^n.$$

- Marcus–Spielman–Srivastava (2022):

$$(p_n^1 \boxtimes_n p_n^2)(x) = \hat{p}_n^1(-x\frac{d}{dx})\hat{p}_n^2(-x\frac{d}{dx})(x-1)^n, \quad \text{for } p_n^j(x) = \hat{p}_n^j(-x\frac{d}{dx})(x-1)^n.$$

Question: what happens with other **differential operators**?

Any questions?