Orthogonal and Symplectic Integrals via modulated $2j-k$ bi-orthogonal polynomial systems

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The Plan

1. characteristic polynomials $\Lambda_{A_N}(z)$ of a random matrix A_N can model L-functions
2. Large N asymptotics of such characteristic polynomials
3. Determinants of moments with a modulated or slanted structure - not Toeplitz
4. Integrable nature of these particular determinants
5. Practical tools for computing and studying coefficients

Katz & Sarnak provide evidence of 4 families of L-functions with the following symmetry types:

- unitary $\mathrm{U}(N)$,
- unitary symplectic $\mathrm{USp}(2N)$,
- even orthogonal $\mathrm{O}^+(2N)$ and
- odd orthogonal $\mathrm{O}^-(2N)$

L-functions are modelled by characteristic polynomials $\Lambda_A(z)$ of a random matrix A.

[Conrey, Rubinstein & Snaith 2006] developed an efficient method to compute such averages in the unitary case:

$$\int_{\mathrm{U}(N)} |\Lambda_A'(1)|^{2m}\, dA = b_m N^{m^2+2m} + O\left(N^{m^2+2m-1}\right)$$

Explicitly, their coefficient is as follows:

$$b_m = (-1)^m \sum_{h=0}^{m} \binom{m}{h} \left(\frac{d}{dt}\right)^{h+m} \left(e^{-t} t^{-m^2/2} \det_{m\times m}\left(I_{m+j-k}(2\sqrt{t})\right)\right)\bigg|_{t=0}$$

where $I_\ell(x)$ is the modified Bessel function of the first kind.

[Forrester & W 2006] these determinants of I-Bessel functions can be given in terms of solutions to Painlevé III′ differential equations

One then defines

$$\tau_m(t) := 2^{-m(m-1)} t^{-m^2/2} \det_{m\times m}\left(I_{m+j-k}(2\sqrt{t}) \right)$$

[Forrester & W 2002] knew that this $\tau_m(t)$ is in fact the Okamoto τ-function associated with the Painlevé III′ σ-form:

$$(t\sigma'')^2 + \sigma'(4\sigma' - 1)(\sigma - t\sigma') - \tfrac{1}{4}m^2 = 0$$

This nonlinear second order differential equation has a solution with certain boundary data given in terms of $\tau_m(t)$ by the formula

$$\sigma_{\mathrm{III},m}(t) = -t\frac{d}{dt}\log\left(e^{-t/4} t^{m^2} \tau_m\left(\tfrac{1}{4}t\right)\right)$$

Specifying boundary conditions, one can quickly compute $\sigma_{\mathrm{III},m}(t)$ from the differential equation and recover $\tau_m(t)$ via the equation

$$\tau_m(t) = \exp\left(-\int_0^{4t} \frac{ds}{s}(\sigma_{\mathrm{III},m}(s) + m^2 - \tfrac{1}{4}s) \right).$$

This expression allows a much faster computation of the constants b_m than earlier methods.

[Ali Altuğ, Bettin, Petrow, Rishikesh & Whitehead 2014] compute

$$M_m(\mathrm{G}(2N), s) := \int_{\mathrm{G}(2N)} \left(\Lambda_A^{(s)}(1)\right)^m \, dA$$

where G denotes USp, SO, or O^-, and dA is the Haar measure on G.

As $N \to \infty$, this models the mth moment of $L^{(s)}(1/2)$ in a family of symmetry type G

$$\int_{\mathrm{G}(2N)} \left(\Lambda_A^{(s)}(1)\right)^m \, dA = b_m N^{\ell_m} + O\left(N^{\ell_m - 1}\right)$$

s is the differentiation order
b_m is a geometrical constant
ℓ_m is an exponent dependent upon G

For $u \in \mathbb{C}$ and $l \in \mathbb{Z}$ let

$$w_l(u) = \oint_{|z|=1} \frac{e^{z+uz^{-2}}}{z^{l+1}} \frac{dz}{2\pi i} = \frac{1}{\Gamma(l+1)} {}_0F_2\left(; \tfrac{1}{2}l+1, \tfrac{1}{2}(l+1); \tfrac{1}{4}u\right)$$

The role of the τ-function is now played by

$$D_n^{(r)}(u) := \det\left(w_{2j-k+r}(u)\right)_{0 \le j,k \le n-1}, \quad n \ge 0$$

USp: $M_m(\mathrm{USp}(2N), 2) = b_m(\mathrm{USp}(2N), 2) \cdot (2N)^{\frac{m^2+5m}{2}} + O(N^{\frac{m^2+3m}{2}})$ where

$$b_m(\mathrm{USp}(2N), 2) = 2^{-\frac{m^2+5m}{2}} \frac{d^m}{du^m} \left(e^u D_m^{(1)}(2u)\right)\Big|_{u=0}$$

O^+: $M_m(\mathrm{O}^+(2N), 2) = b_m(\mathrm{O}^+(2N), 2) \cdot (2N)^{\frac{m^2+3m}{2}} + O(N^{\frac{m^2+m}{2}})$ where

$$b_m(\mathrm{O}^+(2N), 2) = 2^{-\frac{m^2+m}{2}} \frac{d^m}{du^m} \left(e^u D_m^{(0)}(2u)\right)\Big|_{u=0}$$

O^-: $M_m(\mathrm{O}^-(2N), 3) = b_m(\mathrm{O}^-(2N), 3) \cdot (2N)^{\frac{m^2+5m}{2}} + O(N^{\frac{m^2+3m}{2}})$ where

$$b_m(\mathrm{O}^-(2N), 3) = 3 \cdot 2^{-\frac{m^2+3m}{2}} \frac{d^m}{du^m} \left(e^u D_m^{(1)}(2u)\right)\Big|_{u=0}$$

Consider an arbitrary sequence of moments $\{w_l\}_{l\in\mathbb{Z}}$.

Let $\boldsymbol{D}_n^{(r)}$ denote the $n\times n$ matrices of $2j-k$ structure and denote their determinants by D_n:

$$D_n^{(r)} := \det\begin{pmatrix} w_r & w_{r-1} & \cdots & w_{r-n+1} \\ w_{r+2} & w_{r+1} & \cdots & w_{r-n+3} \\ \vdots & \vdots & \vdots & \vdots \\ w_{r+2n-2} & w_{r+2n-3} & \cdots & w_{r+n-1} \end{pmatrix} \equiv \det_{0\le j,k\le n-1}\left(w_{r+2j-k}\right)$$

Joint density function has the form

$$\prod_{1\le j<k\le n}\left(\zeta_k-\zeta_j\right)\left(\zeta_k^{-2}-\zeta_j^{-2}\right), \quad \zeta_l := e^{i\theta_l}\in\mathbb{T}, \quad \theta_l\in(-\pi,\pi]$$

which is distinct from modulus thereof, i.e.

$$\prod_{1\le j<k\le n}\left|\zeta_k-\zeta_j\right|^2\left|\zeta_k+\zeta_j\right|$$

Recall

$$D_n^{(r)} := \det \begin{pmatrix} w_r & w_{r-1} & \cdots & w_{r-n+1} \\ w_{r+2} & w_{r+1} & \cdots & w_{r-n+3} \\ \vdots & \vdots & \vdots & \vdots \\ w_{r+2n-2} & w_{r+2n-3} & \cdots & w_{r+n-1} \end{pmatrix} \equiv \det_{0\leq j,k\leq n-1} \left(w_{r+2j-k} \right)$$

Definition

For each offset value $r \in \mathbb{Z}$, define the $2j-k$ sequences of monic polynomials $\{P_n(z;r)\}_{n=0}^{\infty}$ and $\{Q_n(z;r)\}_{n=0}^{\infty}$, $\deg P_n(z;r) = \deg Q_m(z;r) = n$, satisfying the *bi-orthogonality* condition:

$$\int_{\mathbb{T}} P_m(\zeta;r) Q_n(\zeta^{-2};r) \zeta^{-r} \frac{d\mu(\zeta)}{2\pi i \zeta} = h_n^{(r)} \delta_{mn}, \qquad m,n \in \mathbb{N} \cup \{0\},$$

where $h_n^{(r)}$ is the *norms* of the polynomials squared and $d\mu(\zeta) \equiv w(\zeta) d\zeta$ for some weight function $w(z)$.

If $D_n^{(r)} \neq 0$, the polynomials $P_n(z;r)$ and $Q_m(z;r)$ exist and are uniquely given by

$$P_n(z;r) = \frac{1}{D_n^{(r)}} \det \begin{pmatrix} w_r & w_{r-1} & \cdots & w_{r-n} \\ w_{r+2} & w_{r+1} & \cdots & w_{r-n+2} \\ \vdots & \vdots & \vdots & \vdots \\ w_{r+2n-2} & w_{r+2n-3} & \cdots & w_{r+n-2} \\ 1 & z & \cdots & z^n \end{pmatrix}$$

Observe $h_n^{(r)} = D_{n+1}^{(r)}/D_n^{(r)}$

Analogues of the Szegö recurrences are the third order n-recurrences

$$P_{n+3}(z;r) + \left(\frac{h_{n+2}^{(r-1)}}{h_{n+2}^{(r)}} + \frac{h_{n+1}^{(r-2)}}{h_{n+1}^{(r-1)}} \right) P_{n+2}(z;r) + \left(\frac{h_{n+1}^{(r-2)}}{h_{n+1}^{(r)}} - z^2 \right) P_{n+1}(z;r) - \frac{h_{n+1}^{(r-2)}}{h_n^{(r)}} z^2 P_n(z;r) = 0$$

and for $Q_n^*(z;r) := z^n Q_n(z^{-1};r)$

$$Q_{n+3}^*(z;r) - \left(1 - \frac{h_{n+2}^{(r+2)}}{h_{n+2}^{(r)}} z \right) Q_{n+2}^*(z;r) - \left(\frac{h_{n+2}^{(r+1)}}{h_{n+1}^{(r)}} + \frac{h_{n+2}^{(r+2)}}{h_{n+1}^{(r+1)}} \right) z Q_{n+1}^*(z;r) - \frac{h_{n+2}^{(r+2)}}{h_n^{(r)}} z^2 Q_n^*(z;r) = 0$$

Third order pure-offset r-recurrence relations for the $2j-k$polynomials are given by

$$\frac{D_n^{(r+3)} D_{n+1}^{(r)}}{D_{n+1}^{(r+1)} D_n^{(r+2)}} P_n(z;r+3) - z P_n(z;r+2) - \frac{h_n^{(r)}}{h_n^{(r+1)}} P_n(z;r+1) + z P_n(z;r) = 0$$

and

$$Q_n^*(z;r+3) - Q_n^*(z;r+2) - \frac{h_n^{(r+3)}}{h_n^{(r+1)}} z Q_n^*(z;r+1) + \frac{D_n^{(r)} D_{n+1}^{(r+3)}}{D_{n+1}^{(r+1)} D_n^{(r+2)}} z Q_n^*(z;r) = 0$$

Let

$$P_n(z;r) = z^n + \sum_{j=0}^{n-1} p_{n,j}^{(r)} z^{n-j}, \qquad Q_n(z;r) = z^n + \sum_{j=0}^{n-1} q_{n,j}^{(r)} z^{n-j}$$

Two pure n-recurrence relations for the sub-leading coefficients

$$p_{n+2,1}^{(r)} - p_{n,1}^{(r)} = -\frac{h_{n+1}^{(r-1)}}{h_{n+1}^{(r)}} - \frac{h_n^{(r-2)}}{h_n^{(r-1)}} + \frac{h_n^{(r-2)}}{h_{n-1}^{(r)}}, \quad q_{n+1,1}^{(r)} - q_{n,1}^{(r)} = -\frac{h_n^{(r+2)}}{h_n^{(r)}} + \frac{h_n^{(r+1)}}{h_{n-1}^{(r)}} + \frac{h_n^{(r+2)}}{h_{n-1}^{(r+1)}}$$

Two pure r-recurrences

$$p_{n,1}^{(r+2)} - p_{n,1}^{(r)} = -\frac{h_n^{(r)}}{h_{n-1}^{(r+2)}}, \quad q_{n,1}^{(r+1)} - q_{n,1}^{(r)} = \frac{h_n^{(r+1)}}{h_{n-1}^{(r)}}$$

and in total there are 4 mixed versions such as

$$q_{n+1,1}^{(r)} - q_{n,1}^{(r+2)} = -\frac{h_n^{(r+2)}}{h_n^{(r)}}$$

Reproducing kernel is defined by

$$K_n(z,\zeta;r) := \sum_{j=0}^{n} \frac{1}{h_j^{(r)}} Q_j(z;r) P_j(\zeta;r)$$

The Christoffel-Darboux identity for the "$2j-k$" system can be written as

$$K_n(z_2,z_1;r) = \frac{1}{z_2 z_1^2 - 1} \Bigg\{ \frac{1}{h_n^{(r+2)}} P_n(z_1;r+2) Q_{n+1}(z_2;r) + \frac{1}{h_n^{(r)}} z_2 P_{n+2}(z_1;r+2) Q_n(z_2;r)$$
$$+ \frac{1}{h_{n+1}^{(r+2)}} P_{n+1}(z_1;r+2) \left[Q_{n+2}(z_2;r) - \left(z_2 + q_{n+2,1}^{(r)} - q_{n+1,1}^{(r+2)} \right) Q_{n+1}(z_2;r) \right] \Bigg\}$$

Recall

$$w(z;u) = e^{z+uz^{-2}}$$

Pearson-type relation $w' := \partial_z w$

$$\frac{w'}{w} = \left(1 - 2uz^{-3}\right)$$

Spectral derivatives

$$zP'_n(z;r) = P_{n+1}(z;r) + \left(n + p^{(r)}_{n,1} - p^{(r)}_{n+1,1} - z\right) P_n(z;r) + 2u\,\frac{h^{(r)}_n}{h^{(r)}_{n-1}} P_{n-1}(z;r)$$

and

$$zQ'_n(z;r) = uQ_{n+1}(z;r) + \left[n + u\left(q^{(r)}_{n,1} - q^{(r)}_{n+1,1} - z\right)\right] Q_n(z;r) + \frac{1}{2}\,\frac{h^{(r)}_n}{h^{(r)}_{n-1}} Q_{n-1}(z;r)$$

Considering

$$\int_{\mathbb{T}} P_n(\zeta;r)Q_m(\zeta^{-2};r)\zeta^k\left(1-2u\zeta^{-3}\right)w(\zeta;u)\zeta^{-r}\frac{\mathrm{d}\zeta}{2\pi\mathrm{i}\zeta}$$

it is possible to deduce a sequence of identities $k=\pm 1,3,\dots$ for $m-n=0,\pm 1,\dots$

E.g. for $k=1$, $m=n$:

$$p_{n,1}^{(r)}-2uq_{n,1}^{(r)}=-nr-\tfrac{1}{2}n(n-1)$$

Pearson-type relation $\dot{w} := \partial_u w$

$$\frac{\dot{w}}{w} = z^{-2}$$

Linear differential-difference equations for the weight

$$\partial_u w_r = w_{r+2}$$

Third order linear differential equation

$$4u^2 \partial_u^3 w_r + (4r+10) u \partial_u^2 w_r + (r+1)(r+2) \partial_u w_r - w_r = 0$$

Third order linear difference equation

$$2u w_{r+3} + (r+1) w_{r+1} - w_r = 0$$

Polynomial deformation derivatives are:

$$\dot{P}_n(z;r) = -\frac{h_n^{(r)}}{h_{n-1}^{(r)}} P_{n-1}(z;r)$$

$$\dot{Q}_n(z;r) = Q_{n+1}(z;r) + Q_n(z;r) \left(q_{n,1}^{(r)} - q_{n+1,1}^{(r)} - z \right)$$

Define the rank 3 system polynomial variables, all with assumed offset r

$$\mathcal{P}_n(z;u) := \begin{pmatrix} P_{n+1}(z;u) \\ P_n(z;u) \\ P_{n-1}(z;u) \end{pmatrix}, \qquad \mathcal{Q}_n(z;u) := \begin{pmatrix} Q_{n+1}(z;u) \\ Q_n(z;u) \\ Q_{n-1}(z;u) \end{pmatrix}$$

N.B. Other choices of linearly-independent triples are possible.

Transfer operators

$$\mathcal{P}_{n+1} = \mathcal{M}_n \mathcal{P}_n, \qquad \mathcal{Q}_{n+1} = \mathcal{N}_n \mathcal{Q}_n$$

with the transfer matrices given by

$$\mathcal{M}_n(z;r) = z^2 \begin{pmatrix} 0 & 1 & \frac{h_n^{(r-2)}}{h_{n-1}^{(r)}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -\frac{h_{n+1}^{(r-1)}}{h_{n+1}^{(r)}} - \frac{h_n^{(r-2)}}{h_n^{(r-1)}} & -\frac{h_n^{(r-2)}}{h_n^{(r)}} & \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

N.B. Another discrete structure is the offset shift $r \mapsto r+1$.

The spectral derivatives are denoted

$$\partial_z \mathcal{P}_n = \mathcal{A}_n \mathcal{P}_n, \qquad \partial_z \mathcal{Q}_n = \mathcal{U}_n \mathcal{Q}_n$$

The spectral matrices are

$$\mathcal{A}_n(z;r) = z\begin{pmatrix} 0 & 1 & \frac{h_n^{(r-2)}}{h_{n-1}^{(r)}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$+\frac{1}{z}\begin{pmatrix} n+1+p_{n+1,1}^{(r)}-p_{n+2,1}^{(r)}-\frac{h_{n+1}^{(r-1)}}{h_{n+1}^{(r)}}-\frac{h_n^{(r-2)}}{h_n^{(r-1)}} & 2u\frac{h_{n+1}^{(r)}}{h_n^{(r)}}-\frac{h_n^{(r-2)}}{h_n^{(r)}} & 0 \\ 1 & n+p_{n,1}^{(r)}-p_{n+1,1}^{(r)} & 2u\frac{h_n^{(r)}}{h_{n-1}^{(r)}} \\ 0 & 1 & n-1+p_{n-1,1}^{(r)}-p_{n,1}^{(r)}-2u\frac{h_{n-1}^{(r)}}{h_{n-1}^{(r-2)}} \end{pmatrix}$$

$$+\frac{1}{z^3}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2u\frac{h_{n-1}^{(r)}}{h_{n-1}^{(r-2)}} & 2u\left(\frac{h_{n-1}^{(r)}}{h_{n-1}^{(r-1)}}+\frac{h_{n-1}^{(r)}}{h_{n-1}^{(r-2)}}\frac{h_n^{(r-1)}}{h_n^{(r)}}\right) & 2u \end{pmatrix}$$

N.B. Two irregular singularities at $z = 0, \infty$ with Poincaré ranks 2

The deformation derivatives are denoted by

$$\partial_u \mathcal{P}_n = \mathcal{B}_n \mathcal{P}_n, \qquad \partial_u \mathcal{Q}_n = \mathcal{V}_n \mathcal{Q}_n$$

The deformation matrices are

$$\mathcal{B}_n(z;r) = \frac{1}{z^2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{h^{(r)}_{n-1}}{h^{(r-2)}_{n-1}} & -\frac{h^{(r)}_{n-1}}{h^{(r-2)}_{n-1}}\frac{h^{(r-1)}_{n}}{h^{(r)}_{n}} - \frac{h^{(r)}_{n-1}}{h^{(r-1)}_{n-1}} & -1 \end{pmatrix} + \begin{pmatrix} 0 & -\frac{h^{(r)}_{n+1}}{h^{(r)}_{n}} & 0 \\ 0 & 0 & -\frac{h^{(r)}_{n}}{h^{(r)}_{n-1}} \\ 0 & 0 & \frac{h^{(r)}_{n-1}}{h^{(r-2)}_{n-1}} \end{pmatrix}$$

Compatibility & Integrability relations of the dynamical system

$$\mathcal{A}_{n+1} \cdot \mathcal{M}_n - \mathcal{M}_n \cdot \mathcal{A}_n = \partial_z \mathcal{M}_n$$

$$\mathcal{B}_{n+1} \cdot \mathcal{M}_n - \mathcal{M}_n \cdot \mathcal{B}_n = \partial_u \mathcal{M}_n$$

$$\mathcal{A}_n \cdot \mathcal{B}_n - \mathcal{B}_n \cdot \mathcal{A}_n = \partial_z \mathcal{B}_n - \partial_u \mathcal{A}_n$$

The σ-function

$$\partial_u \log D_n^{(r)} = -q_{n,1}$$

$P - Q$ Linkage

$$p_{n,1}^{(r)} - 2uq_{n,1}^{(r)} = -nr - \tfrac{1}{2}n(n-1)$$

Closure identities

$$2u\left[\frac{h_{n+1}^{(r)}}{h_n^{(r-1)}} + \frac{h_{n+1}^{(r-1)}}{h_n^{(r-2)}}\right] + \frac{h_{n+1}^{(r-2)}}{h_n^{(r)}} = 2n+2$$

First derivatives

$$\partial_u \log h_n^{(r)} = \frac{h_n^{(r)}}{h_n^{(r-2)}} - \frac{h_{n+1}^{(r-1)}}{h_n^{(r-2)}} - \frac{h_{n+1}^{(r)}}{h_n^{(r-1)}}$$

$$u\partial_u q_{n,1}^{(r)} + q_{n,1}^{(r)} + \tfrac{1}{2}\frac{h_n^{(r)}}{h_{n-1}^{(r)}} = 0$$

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