# Asymptotics of biorthogonal polynomials related to Muttalib–Borodin ensemble and Hermitian random matrix with external source

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Based on joint work with Shuai-Xia Xu ([arXiv:2501.16746] and work in progress), and also previous joint work with Peter Forrester and Lun Zhang

# Definition of biorthogonal polynomials

- Recall the well-known orthogonal polynomials $\{p_n\}_{n=0}^{\infty}$ with respect to a weight function $W(x)$

$$\int p_j(x)p_k(x)W(x)dx = \delta_{j,k}h_k,$$

  where $p_n$ is a monic polynomial of degree $n$, and the integral is assumed to be over $\mathbb{R}$ or a subset of it.

- The **biorthogonal polynomials** $\{p_n\}_{n=0}^{\infty}$ and $\{q_n\}_{n=0}^{\infty}$ with respect to a weight function $W(x)$ and a function $f(x)$, are monic polynomials of degree $n$, such that

$$\int p_j(x)q_k(f(x))W(x)dx = \delta_{j,k}h_k.$$

  If $f(x) = x$, they degenerates into orthogonal polynomials.

- If the weight function $W(x)$ is positive and $f(x)$ is strictly increasing, the biorthogonal polynomials are uniquely defined, like orthogonal polynomials.

# Biorthogonal ensemble

We consider the particle system, such that $n$ particles are distributed on $\mathbb{R}$ or a subset of it, with the joint probability density

$$Z_n^{-1} \prod_{1 \le i < j \le n} (x_i - x_j)(f(x_i) - f(x_j)) \prod_{i=1}^{n} W(x_i).$$

- This is a generalized log-gas model, as there are two types of 2-particle interactions, one is the logarithmic intercation $x_i - x_j$ and the other is modified by $f$. If $f(x) = x$, this is the standard log-gas model, or rather the "orthogonal polynomial ensemble".
- The **biorthogonal ensemble** is a special determinantal point process: The $k$-point correlation functions of the particles, $(k = 1, 2, \dots, n)$, are expressed in a determinantal form by a **correlation kernel** $K_n(x, y)$. Furthermore, the correlation kernel can be expressed by the biorthogonal polynomials with respect to $W(x)$ and $f(x)$:

$$K_n(x, y) = \sum_{j=0}^{n-1} \frac{p_j(x) q_j(f(y))}{h_j} W(x).$$

# Difference from orthogonal polynomial ensemble

- The correlation kernel $K_n(x, y)$ is not symmetric, and probably cannot be symmetrized.
- Generally there is no Christoffel-Darboux formula to simplify the summation formula of $K_n(x, y)$. In some cases, the only way to evaluate the limit of $K_n(x, y)$ is to compute the limits of $p_j(x)$ and $p_j(x)$ for all $j = 0, 1, \ldots, n-1$, and then sum up their products. Although seemingly cumbersome, it often works.

# Muttalib-Borodin (MB) ensemble as a special biorthogonal ensemble

Consider $n$ particles on $\mathbb{R}_+$ with joint pdf

$$Z_n^{-1} \prod_{1 \le i < j \le n} (x_i - x_j)(x_i^\theta - x_j^\theta) \prod_{i=1}^{n} x_i^\alpha e^{-nV(x_i)}, \quad x_i \in \mathbb{R}_+.$$

This is the biorthogonal ensemble with $f(x) = x^\theta$ and $W(x) = x^\alpha e^{-nV(x)}$.

- If $\theta = 1$ and $V(x) = x$, then the MB ensemble becomes the celebrated Wishart ensemble, that is central in multivariate statistics.
- $\theta = 1$: Random Hermitian matrix model of Laguerre type.

$$e^{-n \operatorname{Tr} V(M)}.$$

- $V(x) = x$ was Muttalib's original definition (toy model for quasi-1D conductor) [Muttalib 95]. Later it was connected to random matrices. This distribution can be realized as singular values of an random upper triangular matrix [Cheliotis 18].

# MB ensemble with $V(x) = x$

- Borodin found an explicit formula for the correlation kernel, and found its limit [Borodin 99]. Actually, Borodin introduced the term "biorthogonal ensemble" in this paper.
- [Forrester-W 15] and independently [Zhang 13] expressed the biorthogonal polynomials in contour integral form, and the correlation kernel in double contour integral form. They rederived Borodin's result.
- The biorthogonal polynomials are

$$p_k(x) \sim \frac{e^{nx}}{2\pi i} \oint \Gamma(z+1) \prod_{j=1}^{k} (z - \theta(j-1) - \alpha) \frac{dz}{(nx)^{z+1}},$$

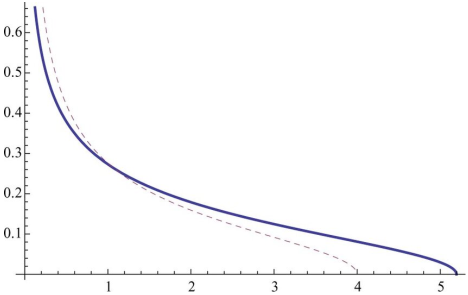
$$q_k(x^\theta) \sim \frac{x^{-\alpha}}{2\pi i} \oint \frac{(nx)^w}{\Gamma(w+1) \prod_{j=1}^{k+1} (w - \theta(j-1) - \alpha)} dw.$$

The correlation kernel formula is

$$K_n(x,y) = \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{x^{-1}(nx)^{-z}\Gamma(z+1) \prod_{k=1}^{n} (z - \theta(j-1) - \alpha)}{(z-w)(ny)^{-w}\Gamma(w+1) \prod_{k=1}^{n} (w - \theta(j-1) - \alpha)}.$$

# Limiting density of the larticles, aka equilibrium measure

The limiting density is given by the Fuss-Catalan distribution with parameter $\theta^{-1}$ (figures from [Claeys-Romano15], with the solid for $\theta = 2$ and dashed for $\theta = 1$, the Marcenko-Pastur law):




- As $x \to 0_+$, the density blows up like $x^{-1/(\theta+1)}$.
- The Fuss-Catalan distribution also occurs in the limiting density of the product of Ginibre matrices, as studied by Akemann, Wei, Kieberg, Ipsen, Forrester, Liu, Kuijlaars, Zhang, Strahov, Burda, Janik, Waclaw, Penson, Zyczkowski, etc. Omission is due to my ignorance.

# Limit of the biorthogonal polynomials at 0 with $\theta \in \mathbb{Z}_+$

With $\rho$ a constant,

- $$p_n\left(\frac{z}{(\rho n)^{1+1/\theta}}\right) \sim \psi(z) + \mathcal{O}\left(n^{-\frac{1}{2\theta+1}}\right),$$
  $$q_n\left(\frac{z^\theta}{(\rho n)^{\theta+1}}\right) \sim \tilde{\psi}(z) + \mathcal{O}\left(n^{-\frac{1}{2\theta+1}}\right),$$

  where the limit functions are given in Meijer G-functions

  $$\psi(z) = z^{\theta-\alpha-1} G^{\theta,0}_{0,\theta+1}\left( {- \atop \frac{\alpha-\theta+1}{\theta}, \frac{\alpha-\theta+2}{\theta}, \ldots, \frac{\alpha-1}{\theta}, \frac{\alpha}{\theta}, 0} \middle| z^\theta \right),$$
  $$\tilde{\psi}(z) = G^{1,0}_{0,\theta+1}\left( {- \atop 0, \frac{-\alpha}{\theta}, \frac{1-\alpha}{\theta}, \ldots, \frac{\theta-1-\alpha}{\theta}} \middle| z^\theta \right).$$

# A brief review of Meijer G-functions

$$G^{m,n}_{p,q}\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| z\right) = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j+u)\prod_{j=1}^{n}\Gamma(1-a_j-u)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-u)\prod_{j=n+1}^{p}\Gamma(a_j+u)} z^{-u} du,$$

where $\Gamma$ denotes the usual gamma function and the branch cut of $z^{-u}$ is taken along the negative real axis.

When $\theta = 1$, both of $\psi(z)$ and $\tilde{\psi}(z)$ become the same Bessel function.

# Limit of the correlation kernel at 0 with $\theta \in \mathbb{Z}_+$

- $$\lim_{n\to\infty}(\rho n)^{-(1+\frac{1}{\theta})}K_n\left(\frac{x}{(\rho n)^{1+1/\theta}}, \frac{x}{(\rho n)^{1+1/\theta}}\right) = K^{(\alpha,\theta)}(x,y),$$

  where

  $$K^{(\alpha,\theta)}(x,y) = \theta^2\int_0^1 (ux)^\alpha \psi(ux)\tilde{\psi}(uy)du$$
  $$= \theta x^{-1}\int_{-i\infty}^{i\infty} ds \oint_{\gamma_-} dt \frac{\Gamma(t)\prod_{i=1}^{\theta}\Gamma(\frac{\alpha+i}{\theta}-s)}{\Gamma(s)\prod_{j=1}^{\theta}\Gamma(\frac{\alpha+j}{\theta}+t)} \frac{x^{\theta s}}{y^{\theta t}} \frac{1}{s-t}.$$

  When $\theta = 1$, it specializes into the Bessel kernel, the universal hard-edge correlation kernel for Laguerre-type matrix models.

# Hard edge universality at 0, $\theta \in \mathbb{Z}_+$

We expect the local universality of the MB ensemble, like in the usual log gas models and random matrix models that are related to orthogonal polynomials (and in many other scenarios beyond orthogonal polynomials, log gas, or random matrix theory). The conjecture is that for any potential $V$, as long as the equilibrium measure determined by $V$ has $x^{1/(\theta+1)}$ blowup behaviour at 0, the limit distribution of the left-most particles is the same as in the $V(x) = x$ special case.

- This local universality is proved in [W-Zhang 22] for $\theta \in \mathbb{Z}_+$ under the technical assumption that $V(x)$ is real analytic and $xV(x)$ is convex (which implies that the equilibrium measure is "one-cut regular").
- The proof is based on a vector Riemann-Hilbert problem (RHP), a variation of the RHP for orthogonal polynomials.
- The technical difficulty is the construction of a $(\theta + 1) \times (\theta + 1)$ model RHP. This construction is explicit, by Meijer G-functions, inspired by the known limiting formulas of $p_n$ and $q_n$.

- The hard edge universality is also proved for $\theta = 1/m$ by [Kuijlaars-Molag 19], [Molag 21], which is essentially equivalent to the $\theta = m$ case due to the change of variable formula

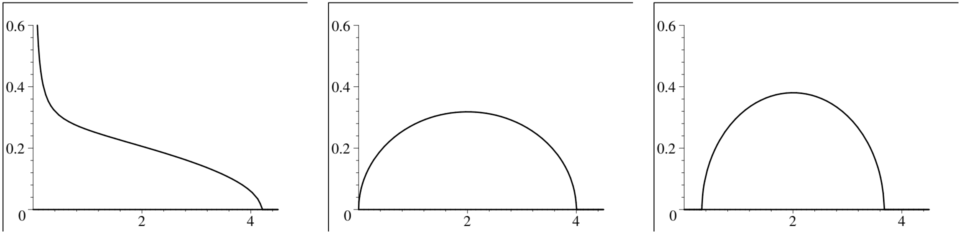
$$\prod_{1\le i<j\le n}(x_i - x_j)(x_i^\theta - x_j^\theta)\prod_{i=1}^{n} x_i^\alpha e^{-nV(x_i)}dx_i$$

$$\Leftrightarrow \prod_{1\le i<j\le n}(y_i^{\frac{1}{\theta}} - y_j^{\frac{1}{\theta}})(y_i - y_j)\prod_{i=1}^{n} y_i^{\frac{\alpha+1}{\theta}-1} e^{-nV(y_i^{1/\theta})}dy_i.$$

- Main difference of the two approaches: [Kuijlaars-Molag 19] and [Molag 21] make use of a Christoffel-Darboux formula and encode it into a matrix-valued RHP, thanks to the equivalence of the MB biorthogonal polynomials with a kind of multiple orthogonal polynomials, and the machinery of large size RHP for multiple orthogonal polynomials (developed by Kuijlaars, Bleher, Van Assche, Aptekarev, etc.) On the other hand, [W-Zhang 22] directly sums up $p_k(x)q_k(y^\theta)$, as mentioned above.

# Hard-to-soft transition for MB ensemble with $\theta = 1$

We consider the limiting density of particles of the MB ensemble, with $V(x)$ that does not satisfy the condition for the hard edge universality. First, we consider the $\theta = 1$ case with $V(x) = \frac{1}{2t}(x-2)^2$ with $t = 1.2$, 1 and 0.7 (Figures from [Claeys-Kuijlaars 08])



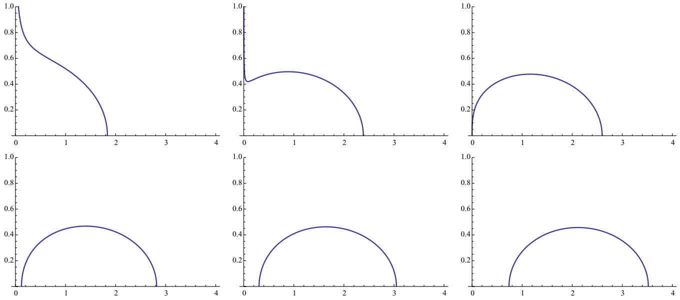

- It is clear that as $t$ decreases to be less than 1, the support of the limiting density is an interval $[a, b]$ with $a > 0$, and the limiting density vanishes like $(x-a)^{1/2}$, a typical **soft edge** behaviour. We naturally expect the Tracy-Widom distribution for the local limit.

- When $t = 1$, the transitioning regime, the support of the limiting density is an interval $[0, 4]$, and the limiting density also vanishes like $x^{1/2}$. However, we expect a different local limiting distribution there.
- It was solved in [Claeys-Kuijlaars 08] that th limiting distribution is described by a $2 \times 2$ RHP associated to the Painlevé XXXIV equation (a variation of the Painlevé II equation).
- The relation between the limiting distribution and the Painlevé equation is by integrable system structure:
  1. First, from the $2 \times 2$ matrix valued RHP for the orthogonal polynomials, a limiting model RHP is derived.
  2. Next, from the Lax pair of the model RHP, we write down the zero-curvature equation, and show that is simplifies into a Painlevé equation.

  (The relation will be explicitly explained later.)

# Hard-to-soft transition for MB ensemble with $\theta = 2$

We consider the limiting density for $\theta = 2$ and $V(x) = x^2 + tx$, with $t = 0, -1.8, -2, -2.5, -3, -4$. We can see that the transition happens at $t = -2$. (Figures from [Claeys-Romano 14].)




In the transition regime, at 0, the density vanishes at the speed of $x^{1/3}$. In general, for $\theta = 2, 3, 4, \ldots$, the density vanishes at the speed of $x^{(\theta-1)/(\theta+1)}$. (But it is not true for $\theta = 1$!)

# Result for the transition regime

In the transition regime, we consider the "double scaling" limit, for example, $\theta = 2$ and $V(x) = x^2 + tx$ with $t = -2 + \tau n^{-1/2}$. With some constant $\rho$,

- $$p_n\left(\frac{z}{(\rho n)^{(\theta+1)/(2\theta)}}\right) \sim \psi^{(\tau)}(z) + \mathcal{O}\left(n^{-\frac{1}{2\theta+1}}\right),$$
  $$q_n\left(\frac{z^\theta}{(\rho n)^{(\theta+1)/2}}\right) \sim \tilde{\psi}^{(\tau)}(z) + \mathcal{O}\left(n^{-\frac{1}{2\theta+1}}\right),$$

  where $\psi^{(\tau)}(z)$ and $\tilde{\psi}^{(\tau)}(z)$ are transcendental functions defined by the model RHP that will be specified later.
- As $\tau \to \infty$, $\psi^{(\tau)}(z)$ (resp. $\tilde{\psi}^{(\tau)}(z)$) converges to $\psi(z)$ (resp. $\tilde{\psi}(z)$) after a scaling transform. As $\tau \to -\infty$, $\psi^{(\tau)}(z)$ (resp. $\tilde{\psi}^{(\tau)}(z)$) converges to the Airy function after a linear transform.

- $$\lim_{n\to\infty}(\rho n)^{-(\theta+1)/(2\theta)}K_n\left(\frac{x}{(\rho n)^{(\theta+1)/(2\theta)}},\frac{x}{(\rho n)^{(\theta+1)/(2\theta)}}\right)=K^{(\alpha,\theta,\tau)}(x,y).$$

  where

  $$K^{(\alpha,\theta,\tau)}(x,y)=\int_{-\infty}^{\tau}\psi^{(\sigma)}(x)\tilde{\psi}^{(\sigma)}(y)d\sigma.$$

### Remark

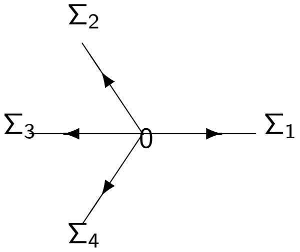
We at last remark that the result for the transitive regime does not hold in the $\theta = 1$ case. The scaling for the transitive regime in the $\theta = 1$ case is $\mathcal{O}(n^{-2/3})$, the same as the soft edge regime, rather than $\mathcal{O}(n^{-1})$ suggested by extrapolating the formulas above.

# Transition model RHP ($\theta = 1$ case)

As constructed in [Claeys-Kuijlaars 08], $\Psi(\zeta)$ is a $2 \times 2$ matrix-valued function, which is analytic for $\zeta$ in $\mathbb{C} \backslash \{\cup_{j=1}^4 \Sigma_j \cup \{0\}\}$, where

1. $\Psi(\zeta)$ satisfies the jump condition

$$\Psi_+(\zeta) = \Psi_-(\zeta) \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \zeta \in \Sigma_1, \\ \begin{pmatrix} 1 & 0 \\ -e^{\alpha\pi i} & 1 \end{pmatrix}, & \zeta \in \Sigma_2, \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, & \zeta \in \Sigma_3, \\ \begin{pmatrix} 1 & 0 \\ -e^{-\alpha\pi i} & 1 \end{pmatrix}, & \zeta \in \Sigma_4. \end{cases}$$



2. As $\zeta \to \infty$,

$$\Psi(\zeta) = \left(I + +\mathcal{O}\left(\zeta^{-1}\right)\right) \frac{\zeta^{-\frac{1}{4}\sigma_3}}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} e^{-(\frac{2}{3}\zeta^{3/2}+\tau\zeta^{1/2})\sigma_3},$$

where the functions $\zeta^{1/4}, \zeta^{1/2}, \zeta^{3/2}$ take the principle branches.

3. As $\zeta \to 0$,

$$\Psi(\zeta) = N(\zeta)\zeta^{\frac{\alpha}{2}\sigma_3}E,$$

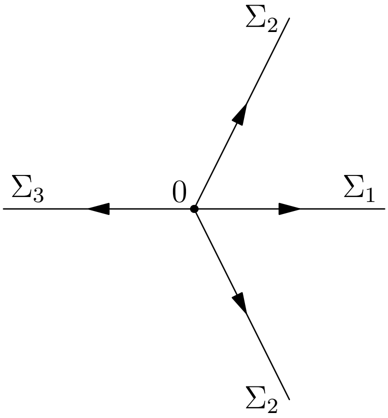
where $E$ is a piecewise constant function on $\mathbb{C} \setminus \Sigma$ if $\alpha \notin \mathbb{Z}$.

# Transition model RHP ($\theta = 2$ case)

$\Psi(\zeta)$ is a $3 \times 3$ matrix-valued function, which is analytic for $\zeta$ in $\mathbb{C}\setminus\{\cup_{j=1}^{4}\Sigma_j \cup \{0\}\}$, where

1. $\Psi(\zeta)$ satisfies the jump condition $\Psi_+(\zeta) = \Psi_-(\zeta)J^{(2)}(\zeta)$, where

$$J^{(2)}(\zeta) = \begin{cases} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \zeta \in \Sigma_1, \\ \begin{pmatrix} 1 & e^{-\frac{2\pi i\alpha}{3}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \zeta \in \Sigma_2, \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, & \zeta \in \Sigma_3, \\ \begin{pmatrix} 1 & -e^{\frac{2\pi i\alpha}{3}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \zeta \in \Sigma_4. \end{cases}$$



# Transitive model RHP ($\theta = 2$ case)

2. $\Psi(\zeta)$ has the following boundary condition as $\zeta \to \infty$

$$\Psi(\zeta) = (I + \mathcal{O}(\zeta^{-1}))\,\mathrm{diag}(1, -\omega\zeta^{\frac{1}{3}}, \omega^2\zeta^{\frac{2}{3}})$$
$$\times \begin{cases} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \mathrm{diag}\left( e^{-\frac{3}{4}\omega^2\zeta^{\frac{2}{3}} - \tau\omega\zeta^{\frac{1}{3}}}, e^{-\frac{3}{4}\omega\zeta^{\frac{2}{3}} - \tau\omega^2\zeta^{\frac{1}{3}}}, e^{-\frac{3}{4}\zeta^{\frac{2}{3}} - \tau\zeta^{\frac{1}{3}}} \right), & \zeta \in \mathbb{C}_+, \\ \begin{pmatrix} 1 & 1 & 1 \\ \omega & 1 & \omega^2 \\ \omega^2 & 1 & \omega \end{pmatrix} \mathrm{diag}\left( e^{-\frac{3}{4}\omega\zeta^{\frac{2}{3}} - \tau\omega^2\zeta^{\frac{1}{3}}}, e^{-\frac{3}{4}\omega^2\zeta^{\frac{2}{3}} - \tau\omega\zeta^{\frac{1}{3}}}, e^{-\frac{3}{4}\zeta^{\frac{2}{3}} - \tau\zeta^{\frac{1}{3}}} \right), & \zeta \in \mathbb{C}_-. \end{cases}$$

3. $\Psi(\zeta)$ has the following boundary condition as $z \to 0$

$$\Psi(\zeta) = N(\zeta)\,\mathrm{diag}\left(\zeta^{\frac{1}{2}-\frac{\alpha}{3}}, \zeta^{\frac{\alpha}{6}}, \zeta^{\frac{1}{2}+\frac{\alpha}{6}}\right) E,$$

where $E$ is a piecewise constant function on $\mathbb{C} \setminus \Sigma$ if $\alpha \notin \mathbb{Z}$.

# Model RHP for $\theta \in \mathbb{Z}_+$ and $\psi^{(\tau)}(z)$ and $\tilde{\psi}^{(\tau)}(z)$

- For $\theta \in \mathbb{Z}_+$, the model RHP is of size $(\theta + 1) \times (\theta + 1)$.
- $\psi^{(\tau)}(z)$ is represented by the first row of the model RHP.
- $\tilde{\psi}^{(\tau)}(z)$ is represented by the first column of the inverse of the model RHP.

# Lax pair ($\theta = 1$)

From the solution of the transitive model RHP, we define the Lax pair

$$\frac{d}{d\zeta}\Psi = A\Psi, \qquad \frac{d}{d\tau}\Psi = B\Psi.$$

From the structure of the RHP, we conclude that $A, B$ are both analytic in $\zeta \in \mathbb{C} \setminus \{0\}$. We also have that the Laurent series of $A$ has only the degree 1, degree 0 and degree $-1$ terms, and the Laurent series of $B$ has only the degree 1 and degree 0 terms. The Lax pair can be expressed as [Its-Kuijlaars-Östensson 08]

$$\frac{\partial}{\partial\zeta}\Psi(\zeta) = \begin{pmatrix} \frac{u_\tau}{2\zeta} & i - i\frac{u}{\zeta} \\ -i\zeta - i(u+\tau) - i\frac{u_\tau^2 - \alpha^2}{4u\zeta} & -\frac{u_\tau}{2\zeta} \end{pmatrix} \Psi(\zeta),$$

$$\frac{\partial}{\partial\tau}\Psi(\zeta) = \begin{pmatrix} 0 & i \\ -i\zeta - 2i(u + \tau/2) & 0 \end{pmatrix} \Psi(\zeta).$$

# Lax pair ($\theta = 2$)

From the solution of the transitive model RHP, we define the Lax pair

$$\frac{d}{d\zeta}\Psi = A\Psi, \qquad \frac{d}{d\tau}\Psi = B\Psi.$$

From the structure of the RHP, we conclude that $A, B$ are both analytic in $\zeta \in \mathbb{C} \setminus \{0\}$. We also have that the Laurent series of $A$ has only the degree 0 and degree $-1$ terms, and the Laurent series of $B$ has only the degree 1 and degree 0 terms.

$$A = A_0(\tau) + A_{-1}(\tau)\zeta^{-1}, \qquad B = B_1(\tau)\zeta + B_0(\tau),$$

with

$$A_0 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -\tau & 1 & 0 \end{pmatrix}, \qquad A_{-1} = \begin{pmatrix} b & c+\tau & -1 \\ a & -b-f+\frac{1}{3} & -c+\tau \\ d & k & f+\frac{2}{3} \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad B_0 = \begin{pmatrix} -c & 1 & 0 \\ f-\tau c & 0 & 1 \\ \tau(b+f)-(a+k) & b+\tau c & c \end{pmatrix}.$$

# Zero-curvature equation

Then we have the zero-curvature equation for

$$\frac{dA}{d\tau} - \frac{dB}{d\zeta} + AB - BA = 0.$$

- In the $\theta = 1$ case, it implies the Painlevé XXXIV equation

$$u''(s) = 4u(s)^2 + 2su(s) + (2u(s))^{-1}(u'(s)^2 - \alpha^2).$$

- In the $\theta = 2$ case, the zero-curvature equation then becomes

$$\frac{b'}{3} = -cf - k + \tau(b + c^2) + \tau^2 c, \quad \frac{c'}{3} = -c^2 - b - f, \quad \frac{f'}{3} = -bc + a - \tau(f + c^2) + \tau^2 c,$$

$$\frac{a'}{3} = 2bf + f^2 - ck + d - \frac{1}{3}f - \tau(bc - a - k - \frac{1}{3}c) - \tau^2(b + f),$$

$$\frac{k'}{3} = -b^2 - 2bf - ac - d - \frac{1}{3}b - \tau(cf + a + k + \frac{1}{3}c) + \tau^2(b + f),$$

$$\frac{d'}{3} = 2cd - bk + af + \frac{2}{3}a + \frac{2}{3}k - \tau(f^2 - b^2 - ac - ck + \frac{2}{3}b + \frac{2}{3}f).$$

After some algebraic manipulations, the six equations are reduced into one:

$$c''' + 3 \cdot 2^{\frac{3}{2}} c'^2 + \frac{4}{3}\tau^2 c' + 4\tau c + \frac{\sqrt{2}}{9}(1 + 3\alpha - 3\alpha^2) = 0.$$

Let $y(\tau) = c(\frac{\tau}{\sqrt{2}}) + \frac{\tau^3}{108}$, then $y$ satisfies the Chazy-I equation

$$y''' + 6y'^2 + \tau y - \frac{1}{72}\tau^4 + \frac{1}{6}(\alpha - \alpha^2) = 0.$$

Furthermore, let $u(\tau) = \sqrt{2}c(\tau) + \frac{4}{27}\tau^3$, we have

$$(u'')^2 + 4(u')^3 - 4(\tau u' - u)^2 + \frac{4}{3}(\alpha - \alpha^2 - 1)u' + \frac{4}{27}(\alpha + 1)(2\alpha - 1)(\alpha - 2) = 0,$$

which is the third member of the Chazy's system (III) and can be solved in terms of the Painlevé IV equation.

# Relation to Boussinesq equation

Consider the Boussinesq equation

$$u_{tt} + \frac{1}{2}(u^2)_{xx} + u_{xxxx} = 0.$$

If we take the similarity reduction ($\alpha, \beta, \gamma$ are constants)

$$u(x,t) = \frac{g(z)}{t + \gamma/(2\alpha)}, \qquad \tau = \frac{x + \beta/\alpha}{[t + \gamma/(2\alpha)]^{1/2}},$$

then $g(\tau)$ satisfies [Clarkson-Kruskal 89]

$$\frac{\tau^2}{4}\frac{d^2g}{d\tau^2} + \frac{7\tau}{4}\frac{dg}{d\tau} + 2g + g\frac{d^2g}{d\tau^2} + \left(\frac{dg}{d\tau}\right)^2 + \frac{d^2g}{d\tau^4} = 0.$$

This is satisfied by $g(\tau) = 3\sqrt{6}c'(3^{1/4}\tau/2)$.

## Remark

The Painlevé XXXIV equation is related to the KdV equation in a similar way.

# Relation to Drinfeld-Sokolov/Gelfand-Dickey hierarchies

- [Liu-Wu-Zhang 22] studies the similarity reductions of Drinfeld-Sokolov hierarchies. They find that for the hierarchies associated to $A_1^{(1)}$ (KdV) and $A_2^{(1)}$ (Boussinesq), the reductions yield the Lax pairs for the $\theta = 1$ and $\theta = 2$ cases.
- We find that for general $\theta \in \mathbb{Z}_+$, the Lax pairs are related to the Drinfeld-Sokolov hierarchies associated to $A_\theta^{(1)}$, or the Gelfand-Dickey hierarchies, in this way.

# Muttalib-Borodin on the real line with $\theta = 2$

Now we consider the Hermite type, rather than the Lagerre type, of the MB ensemble with $\theta = 2$

## Remark

In [Borodin 99], a Hermite type biorthogonal ensemble is defined. But our definition is different from the one in [Borodin 99].

- We may consider the joint distribution function

$$\prod_{1 \le i < j \le n} (x_i - x_j)(x_i^2 - x_j^2) \prod_{i=1}^{n} W(x_i), \quad x_i \in \mathbb{R},$$

  and similarly we can define the biorthogonal ensembles $p_n$ and $q_n$, and then $K_n(x, y)$.
- But this is **not** a probability density, and the total integral may not be nonzero. This makes the well-definedness of $p_n$, $q_n$, and $K_n(x, y)$ troublesome.

# Well-definedness of a special case

Consider the joint density function for $V(x)$ being an **even** function, $a > 0$ and $\alpha > -1$:

$$\prod_{1\le i<j\le n}(x_i - x_j)(x_i^2 - x_j^2)\prod_{i=1}^{n} W^{(c)}(x_i), \quad W^{(c)}(x) = |x|^\alpha e^{-2n(V(x)-ax)} \times \begin{cases} 1, & x \ge 0, \\ c, & x < 0. \end{cases}$$

Then define the biorthogonal polynomials $p_n, q_n$ by the biorthogonality

$$\langle p_j^{(c)}(x), q_k^{(c)}(x^2)\rangle_c = \delta_{jk} h_j^{(c)}, \quad \langle f(x), g(x)\rangle_c = \int_{\mathbb{R}} f(x)g(x)W^{(c)}(x)dx,$$

We have that if $c \in [-1, 1]$, then the total integral of the joint density function is nonzero, and $p_n^{(c)}$, $q_n^{(c)}$, and $K_n^{(c)}(x, y)$ are well defined.

# Hermitian matrix with symmetric external source model

Denote the $2n$-dimensional diagonal matrix

$$A = \mathrm{diag}(\underbrace{a, \ldots, a}_{n \text{ entries}}, \underbrace{-a, \ldots, -a}_{n \text{ entries}}), \quad a > 0.$$

We consider the $2n$-dimensional random Hermitian matrix $M$ whose distribution is given by

$$\frac{1}{C} |\det(M)|^{\alpha} e^{-2n \,\mathrm{Tr}(V(M) - AM)}$$

where the potential function $V$ is an **even** real analytic function, and $\alpha > -1$. If $\alpha = 0$, this is the well known Hermitian matrix model with external source.

- It is well known that the distribution of the eigenvalues of $M$ is a determinantal point process, with correlation kernel $K_{2n}^{\mathrm{ext}}(x, y)$.
- Too many people have had contributions in this model: Brezin, Hikami, Zinn-Justin, Tracy, Widom, Adler, van Moerbeke, Bleher, Kuijlaars, Aptekarev, Delvaux, McLaughlin, Siva, Martínez-Finkelshtein, Erdös, Krüger, Schröder, .... Sorry for the omission due to my ignorance.

# Relation between the generalized Muttalib-Borodin ensemble and the external source model

We define the reproducing kernel

$$K_m^{(c)}(x,y) = \sum_{k=0}^{m-1} \frac{1}{h_k^{(c)}} p_k^{(c)}(x) q_k^{(c)}(y^2) W^{(c)}(x),$$

and

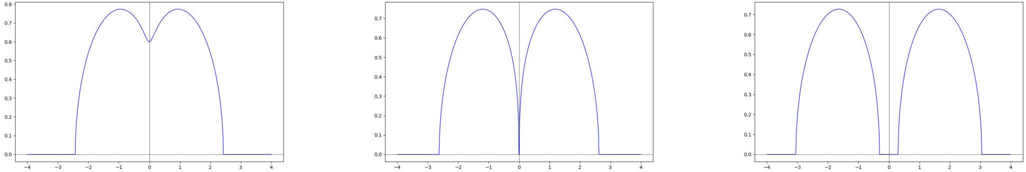
$$\hat{K}_m^{(c)}(x,y) = K_m^{(c)}(x,y) + K_m^{(c)}(-x,y).$$

When $c=0$, we have that $\hat{K}_m^{(0)}(x,y) = K_m^{(0)}(x,y)$ for $x,y>0$ and it is the correlation kernel of the Muttalib-Borodin ensemble.

The correlation kernel for the eigenvalues of the Hermitian matrix model with external source is

$$K_{2n}^{\text{ext}}(x,y) = \frac{1}{2}\hat{K}_n^{(1)}(x,y) + \frac{y}{2x}\hat{K}_n^{(-1)}(x,y)\Big|_{\alpha\to\alpha+1}.$$

# Pearcey limit in the external source model



Suppose $V(x) = x^2/2$, and $a$ increases from 0.8 to 1 and to 1.5. We have that as the limiting density of the eigenvalues of $M$ has a transition, from one-cut to two-cut. At $a = 1$, we observe the transition, and the cusp behaviour is $|x|^{1/3}$.

# A remark on equilibrium measure

For the external source model with quadratic potential, Bleher, Kuijlaars and Aptekarev analyzed it thoroughly in a series of three seminal papers, which initiated the study of random matrices models by Riemann-Hilbert problems of size larger than $2 \times 2$. A key step is to construct a vector equilibrium measure.

In our study of Muttalib-Borodin ensemble, we only have a scalar equilibrium measure, namely the limiting density of particles.

However, in the remark on Page 46 of [Bleher-Kuijlaars, CMP 252 (2004), 43–76], it says:

**Remark in [Bleher-Kuijlaars 04]**

[F]or $a > 1$, . . . [i]t is possible to base the asymptotic analysis of the RH problem on the minimization problem, as done by Deift et al, see [14–16], for the unitarily invariant random matrix model. However, we will not pursue that here.

Here, this approach is pursued!

# Relation between the external source model and MB ensemble in the sense of equilibrium measure

- If $a > 1$, then the generalized MB ensemble with $\theta = 2$, $V(x) = x^2/2$ and external strength $a$ associated to the external source model is asymptotically not different from the Muttalib-Borodin ensemble on $\mathbb{R}_+$. Because the potential $V(x) - ax$ favors the right side, the part of domain $(-\infty, 0)$ is negligible. Hence, we can use the global density of the Muttalib-Borodin ensemble as the equilibrium measure that is foreseen but not pursued in [Bleher-Kuijlaars 04].
- If $a = 1$, this idea still works. It is a tiny extension to Bleher and Kuijlaars's observation, but a key step for us.
- If $0 < a < 1$, the bahaviour of the generalized MB ensemble differs a lot from the Muttalib-Borodin ensemble on $\mathbb{R}_+$. This approach fails. We need to use more advanced techniques like vector equilibrium measure.

# Result that generalizes the Pearcey kernel

- For the generalized Muttalib-Borodin ensemble with $\theta = 2$, $V(x) = x^2/2$ and $a = 1 + 2^{-3/2} n^{-1/2} \tau$, we have the limit of biorthogonal polynomials $p_n^{(c)}(x)$ and $q_n^{(c)}(y^2)$ as

$$p_n^{(c)}\left(\frac{z}{n^{3/4}}\right) \sim \psi^{(c,\tau)}(z), \qquad q_n^{(c)}\left(\frac{z^2}{n^{3/2}}\right) \sim \tilde{\psi}^{(c,\tau)}(z),$$

  where $\psi^{(c,\tau)}(z)$ is expressed by the first row of the $3 \times 3$ model Riemann-Hilbert problem in next slide, and $\tilde{\psi}^{(c,\tau)}(z)$ is expressed by the first column of its inverse.
- By summing up all $p_k^{(c)}(x) p_k^{(c)}(y^2)$, we derive the limiting kernel $K_n^{(c)}(x, y)$ and $\hat{K}_n^{(c)}(x, y)$, and then derive the limiting kernel $K_{2n}^{\text{ext}}(x, y)$ for the external source model. Hence, limiting formulas for $p_k^{(c)}(x) p_k^{(c)}(y^2)$, with $c = 1$ and $-1$,implies the limiting formula of $K_{2n}^{\text{ext}}(x, y)$ in principal.
- Practically, we have a Christoffel-Darboux formula that can reduce lots of work. But there is no time to show it.

# Model RHP for generalized MB ensemble

$\Psi(\zeta)$ is a $3\times 3$ vector-valued function on $\mathbb{C}$ except for $\mathbb{R}$ and rays $\{\arg\zeta = \pm\pi/4\}$ and $\{\arg\zeta = \pm 3\pi/4\}$, all rays oriented outward.

1. $\Psi_+(\zeta) = \Psi_-(\zeta)J_\Psi(\zeta)$, where

$$J_\Psi(\zeta) = \begin{cases} \begin{pmatrix} 1 & e^{-\frac{2\alpha}{3}\pi i} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \arg\zeta = \frac{\pi}{4}, & \begin{pmatrix} 1 & -e^{\frac{2\alpha}{3}\pi i} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \arg\zeta = -\frac{\pi}{4}, \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -ce^{-\frac{\alpha}{3}\pi i} & 1 \end{pmatrix}, \quad \arg\zeta = \frac{3\pi}{4}, & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & ce^{\frac{\alpha}{3}\pi i} & 1 \end{pmatrix}, \quad \arg\zeta = -\frac{3\pi}{4}, \\ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \zeta\in\mathbb{R}_+, & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \zeta\in\mathbb{R}_-. \end{cases}$$

2. $\Psi(\zeta)$ has the boundary conditions similar to that of MB ensemble in transition regime with $\theta = 2$.

# Integrability and relation to Pearcey kernel

- This RHP is a generalization of the $\theta = 2$ model RHP for the MB ensemble on $\mathbb{R}_+$ in the transtion regime, which is the $c = 0$ special case.
- It has almost the same relation to the Boussinesq equation and the Painlevé IV equation.
- When $c = 1$ and $\alpha = 0$, or $c = -1$ and $\alpha = 1$, this RHP can be explicitly comstructed by integrals in the form of

$$\int e^{\frac{s^4}{4} - \tau s^2 - s\zeta} ds,$$

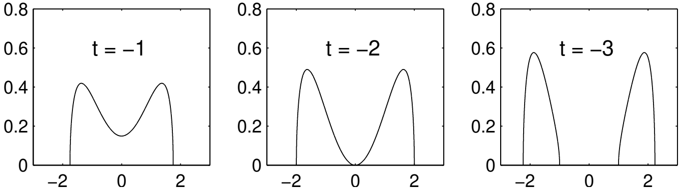
and then the limiting kernel is the Pearcey kernel

$$\frac{1}{(2\pi i)^2} \int ds \int dt \frac{e^{\frac{s^4}{4} - \tau s^2 - sx}}{e^{\frac{t^4}{4} - \tau t^2 - ty}} \frac{1}{s - t}.$$

It is an analogy that a special RHP for Painlevé XXXIV can be constructed by Airy function and yields the Airy kernel.

# When the potential function becomes quartic: $a = 0$ case

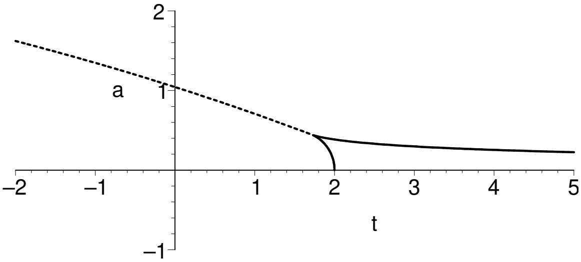
Even without the external source, when the potential function becomes quartic, new phenomenon occurs Suppose the potential is $V(x) = x^4/4 + tx^2/2$ (Figures from [Bleher-Its 03]).




- We again see a transition between one-cut and two-cut, but in the transition regime, the vanishing of density at 0 is faster than the Pearcey transition: it is like $x^2$.
- The local limit is also described by a model RHP, and it is associated to the Painlevé II equation, studied by Bleher, Its, Claeys, Kuijlaars, Vanlessen, etc.
- We naturally expect that this local limit holds for the external source model, at least when $a$ is small.

# The merge of the Pearcey transition and the Painlevé II transition

Consider the external source model with potential $V(x) = x^4/4 - tx^2/2$ and the external source strength $a$. There is a phase diagram (Figure from [Bleher-Delvaux-Kuijlaars 11]). The dotted curve is the Pearcey transition, and the solid curve is the Painlevé II transition.




- We can prove the Pearcey transition like the quadratic potential case.
- We can also compute the limit at the multicritical point where the solid curve and the dotted curve meet.
- Probably we can also prove the Painlevé II transition for the right branch of the solid curve. (It is work in progress.)

# The model RHP for the multicritical limit

As $t = \sqrt{3} + n^{-\frac{1}{2}}\sigma$ and $a = 2 \cdot 3^{-\frac{3}{4}} - n^{-\frac{1}{2}}\sigma + n^{-\frac{3}{4}}\tau$, the limit of the biorthogonal polynomials, and then the limit of the correlation kernel $K^{\text{ext}}_{2n}(x, y)$, is expressed by the model RHP $\Psi(\zeta)$ such that

- $\Psi(\zeta)$ is a $3 \times 3$ vector-valued function on $\mathbb{C}$ except for $\mathbb{R}$ and rays $\{\arg \zeta = \pm\pi/4\}$ and $\{\arg \zeta = \pm 3\pi/4\}$, and is has the same jump condition as the model RHP for the (generalized) Pearcey case. Also its boundary condition at 0 is the same.
-

$$\Psi(\zeta) = (I + \mathcal{O}(\zeta^{-1}))\,\mathrm{diag}(1, -\omega\zeta^{\frac{1}{3}}, \omega^2\zeta^{\frac{2}{3}})$$

$$\times \begin{cases} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \mathrm{diag}\left(e^{\frac{3}{4}\omega\zeta^{\frac{4}{3}} - \sigma\omega^2\zeta^{\frac{2}{3}} - \tau\omega\zeta^{\frac{1}{3}}}, e^{\frac{3}{4}\omega^2\zeta^{\frac{4}{3}} - \sigma\omega\zeta^{\frac{2}{3}} - \tau\omega^2\zeta^{\frac{1}{3}}}, e^{\frac{3}{4}\zeta^{\frac{4}{3}} - \sigma\zeta^{\frac{2}{3}} - \tau\zeta^{\frac{1}{3}}}\right), \zeta \in \mathbb{C}_+, \\ \begin{pmatrix} 1 & 1 & 1 \\ \omega & 1 & \omega^2 \\ \omega^2 & 1 & \omega \end{pmatrix} \mathrm{diag}\left(e^{\frac{3}{4}\omega^2\zeta^{\frac{4}{3}} - \sigma\omega\zeta^{\frac{2}{3}} - \tau\omega^2\zeta^{\frac{1}{3}}}, e^{\frac{3}{4}\omega\zeta^{\frac{4}{3}} - \sigma\omega^2\zeta^{\frac{2}{3}} - \tau\omega\zeta^{\frac{1}{3}}}, e^{\frac{3}{4}\zeta^{\frac{4}{3}} - \sigma\zeta^{\frac{2}{3}} - \tau\zeta^{\frac{1}{3}}}\right), \zeta \in \mathbb{C}_-. \end{cases}$$

# Integrability

This model RHP is associated to the Boussinesq equation and the samilarity reduction of the second member of the Boussinesq Hierarchy:

$$
\begin{aligned}
&u_{\sigma\sigma} = 2v_{\sigma\tau} + u_{\sigma\tau\tau} = -\frac{1}{3}u_{\tau\tau\tau\tau} + \frac{2}{3}(u^2)_{\tau\tau},\\
&u_{\tau\tau\tau\tau} + 2v_{\tau\tau\tau} - (u^2)_{\tau\tau} - 4(uv)_{\tau} + 2\sigma(2v_{\tau} + u_{\tau\tau}) + \tau u_{\tau} + 2u = 0,\\
&\frac{1}{3}\left(2u_{\tau\tau\tau\tau\tau} + 3v_{\tau\tau\tau\tau} - 2(u^2)_{\tau\tau\tau} - 2uu_{\tau\tau\tau} - 6(uv_{\tau})_{\tau} + 6(v^2)_{\tau} + 4u^2u_{\tau}\right)\\
&\qquad + 2\sigma\left(v_{\tau\tau} + \frac{2}{3}u_{\tau\tau\tau} - \frac{1}{3}(u^2)_{\tau}\right) - \tau v_{\tau} - 3v = 0.
\end{aligned}
$$

# Final remark

Technically, the novelty lies in the proof of vanishing lemmas for the model RHPs, so that the solvability of the model RHPs are obtained.

**Thank you for your attention and happy birthday, Peter!**