Non-Hermitian Random Matrix Theories, Integrability and Topology

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Contents

Acknowledgments

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S Matrix Fluctuations

The S matrix describes transitions between two channels which are coupled to the states of an $N \times N$ random matrix Hamiltonian H. We are interested in the large N limit where the number of channels remains fixed. The stochastic S- matrix is given by
Mahaux-Weidenmüller-1969

$$S = \mathbb{1} - 2\pi i w.(E\mathbb{1} - H + \pi i w.w^\dagger)^{-1}.w^\dagger$$

with w the coupling matrix between the states of H and the channels of the S-matrix. The fluctuations of S are universal in terms of $\langle S \rangle$, and are given by a non-linear σ-model. JV-Weidenmüller-Zirnbauer-1983

Note that the poles of the S-matrix are in the lower complex half plane which is implied by causality.

The QCD Dirac Operator at Nonzero Chemical Potential

The Random Matrix Representation of the QCD Dirac operator at nonzero chemical potential μ is given by Jackson-JV-1995, Stephanov-1996, Osborn-2004

$$D = \begin{pmatrix} 0 & iC + \mu B \\ iC^\dagger + \mu B^\dagger & 0 \end{pmatrix}$$

with B and C Gaussian distributed complex random matrices (non-Hermitian RMT ensemble AIII†. For $\mu = 0$, this ensemble is known as the chGUE (or AIII) The matrix D has the chiral symmetry.

$$\Gamma_5 D \Gamma_5 = -D,$$

so that the nonzero eigenvalues are paired as $\pm\lambda$. The spectral density around zero on the scale of the average level spacing is universal, and is described by nonlinear σ model (chiral Lagrangian in high energy lingo). Kogut-Stephanov-Toublan-JV-Zhitnisky-2000

The Wilson QCD Dirac Operator

Random Matrix representation of the Wilson Dirac operator

Damgaard-Splittorff-JV-2010, Akemann-Damgaard-Splittorff-JV-2010, Kieburg-JV -Zafeiropoulos-2011

$$D_W = \begin{pmatrix} aA & iC \\ iC^\dagger & aB \end{pmatrix}, \qquad A = A^\dagger, \quad B = B^\dagger.$$

Here, A and B are GUE and for $a = 0$. D_W is known as the non-hermitian ensemble AIII. The lattice spacing is a.

This Dirac operator is pseudo-Hermitian, $(\Gamma_5 D_W)^\dagger = \Gamma_5 D_W$ with $\Gamma_5 = \mathrm{diag}(1, \cdots, 1, -1, \cdots, -1)$. This has important consequences for the dynamics of the eigenvalues.

The Non-Hermitian Sachdev-Ye-Kitaev Model

This model is defined by the Hamiltonian

$$H = \sum (iM_{ijkl}\psi_i^L\psi_j^L\psi_k^L\psi_l^L - iM_{ijkl}\psi_i^R\psi_j^R\psi_k^R\psi_l^R) + \mu \left(i\sum_k \psi_k^L\psi_k^R \right)^r .$$

Maldacena-Qi-2018, García-García-Nosaka-Rosa-JV-2019, García-García-Jia-Rosa-JV-2021, García-García-Sá-JV-Yin-2024

Here the N operators ψ_k^L and the N operators ψ_k^R are a $2N$ dimensional Clifford algebra, and the matrix elements M_{ijkl} are Gaussian distributed.

This model is solvable for $N \to \infty$, and can be solved analytically when the four-fermion terms are replaced by two-fermion terms.

There is a vast literature on non-Hermitian physics, see review Ashida-Gong-Ueda-2000, even on the Ginibre ensemble, see review Byun-Forrester-2025

II. Integrability at Nonzero Chemical Potential

Spectral Density

Darboux Recursion

Factorization

P.J. Forrester and N.S. Witte, Application of the τ -function theory of Painlevé equations to random matrices: PV , PIII , the LUE, JUE and CUE, Comm. Pure App. Math. 55 (2002) 679.
E. Kanzieper, Replica Field Theories, Painlevé Transcendents, and Exact Correlation Functions, Phys. Rev. Lett. 89 (2002) 250201.

Integrability at Nonzero Chemical Potential

We consider the spectral density of the random matrix model for QCD at nonzero chemical potential. The spectral density in the complex plane is given by

$$\begin{aligned}\rho(z,z^*) &= \sum_k \delta^2(z-\lambda_k) = \frac{1}{\pi}\sum_k \frac{d}{dz^*}\frac{1}{z-\lambda_k} \\ &= \frac{1}{\pi}\frac{d}{dz^*}\lim_{n\to 0}\frac{1}{n}\frac{d}{dz}\log\left(\prod_k (z-\lambda_k)^n (z^*-\lambda_k^*)^n\right)\end{aligned}$$

We conclude that the spectral density can be written as

$$\rho(z,z^*) = \lim_{n\to 0}\frac{1}{\pi n}\frac{d}{dz^*}\frac{d}{dz}\log\left(\det{}^n(z-D)\det{}^n(z^*-D^\dagger)\right).$$

We will analyze this partition function in the microscopic limit, where zN, z^*N and $\mu^2 N$ are kept fixed for $N\to\infty$ (weak non-Hermiticity, (Fyodorov-Khoruzhenko-Sommers-1997)).

Weak Non-Hermiticity Limit

In this limit, the partition function id determined by the symmetries and their spontaneous breaking, $U(2n) \to U(n) \times U(n)$. Itg is give by an integral over the coset $U(2n)/U(n) \times U(n)$, Splittorff-JV-2003

$$Z_n(z, z^*, \mu) = \mathcal{N} \int_{U(2n)/U(n)\times U(n)} dU e^{-N\mu^2 \mathrm{Tr} U B U^\dagger B + \frac{1}{2} N \mathrm{Tr}(MU + MU^\dagger)}$$

with

$$B = \mathrm{diag}(\underbrace{1, \cdots 1}_{n} \underbrace{-1, \cdots - 1}_{n}), \qquad M = \mathrm{diag}(\underbrace{z, \cdots, z}_{n}, \underbrace{z^*, \cdots, z^*}_{n})$$

The integrand does not depend on $U(n) \times U(n)$ and we replace the integral by an integral over $U(2n)$.

Hänkel Form

With some work, the integral can be written in a Hänkel form

$$Z_n(z, z^*, \mu) = \frac{D_n}{(zz^*)^{n(n-1)}} \det \left[\delta_z^k \delta_{z^*}^l Z_1(z, z^*, \mu)\right]_{k,l=0,1,\cdots,n-1}$$

with

$$\delta_z = z\frac{d}{dz}, \quad \delta_{z^*} = z^*\frac{d}{dz^*}.$$

and

$$Z_1(z, z^*, \mu) \quad = \quad \frac{1}{\pi} e^{2N\mu^2} \int_0^1 d\lambda \lambda e^{-2N\mu^2\lambda^2} I_0(\lambda z N) I_0(\lambda z^* N).$$

Determinants of this type were first considered by Darboux in 1888.

Darboux Recursion for Hänkel Determinant

$$
\mathrm{H}_p = \begin{vmatrix} \alpha & \dfrac{\partial\alpha}{\partial x} & \dfrac{\partial^2\alpha}{\partial x^2} & \cdots & \dfrac{\partial^p\alpha}{\partial x^p} \\ \dfrac{\partial\alpha}{\partial y} & \dfrac{\partial^2\alpha}{\partial x\,\partial y} & \cdots & \cdots & \dfrac{\partial^{p+1}\alpha}{\partial x^p\,\partial y} \\ \cdots & \cdots\cdots & \cdots & \cdots & \cdots\cdots\cdots \\ \dfrac{\partial^p\alpha}{\partial y^p} & \cdots\cdots & \cdots & \cdots & \dfrac{\partial^{2p}\alpha}{\partial x^p\,\partial y^p} \end{vmatrix}.
$$

On est donc conduit à l'identité

$$
(10) \qquad \mathrm{H}_{p-1}\mathrm{H}_{p+1} = \mathrm{H}_p \frac{\partial^2 \mathrm{H}_p}{\partial x\,\partial y} - \frac{\partial \mathrm{H}_p}{\partial x}\frac{\partial \mathrm{H}_p}{\partial y},
$$

d'où découle la relation (9) que nous voulions vérifier.

Gaston Darboux, 1842-1917

Gaston Darboux, Leçons sur la théorie générale des surfaces et les applications geometriques du calcul infinitésimal, Deuxième Partie, Gauthier-Villars, Paris, 1889, Livre IV, Ch. VI, pp. 135-149.

Toda Lattice Equation

His analysis is also valid in this case and leads to the recursion relation

$$\delta_z \delta_{z^*} \log Z_n(z, z^*, \mu) = \frac{\pi n}{2}(zz^*)^2 \frac{Z_{n+1}(z, z^*, \mu) Z_{n-1}(z, z^*, \mu)}{Z_n^2(z, z^*, \mu)}.$$

This is the Toda lattice equation (Darboux recursion would be a more appropriate name). It can be proved using the Sylvester identity (see exercise in Peter's Book).

Spectral Density

For the spectral density we obtained

$$\rho(z,z^*) = \lim_{n\to 0} \frac{1}{\pi n} \frac{d}{dz^*} \frac{d}{dz} \log Z_n(z, z^* \mu).$$

Using the Darboux recursion we find

$$\rho(z,z^*) = \frac{\pi}{2}(zz^*)^2 \lim_{n\to 0} \frac{Z_{n+1}(z,z^*,\mu) Z_{n-1}(z,z^*,\mu)}{Z_n^2(z,z^*,\mu)}.$$

Since $Z_0(z,z^*,\mu) = 1$ we obtain the compact equation

$$\rho(z,z^*) = \frac{\pi}{2}(zz^*)^2 Z_1(z,z^*,\mu) Z_{-1}(z,z^*,\mu).$$

Splittorff-JV-2003

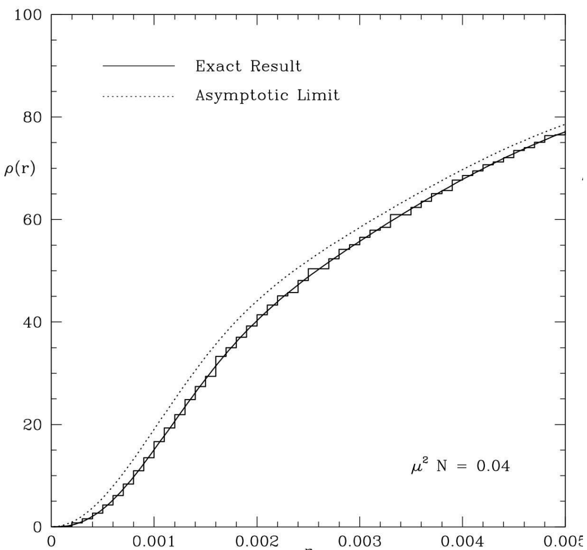
The $Z_{-1}(z,z^*,\mu)$ partition function follows from the symmetry breaking pattern $Gl(4) \to U(2) \times U(2)$. The integral over the noncompact manifold can be worked out analytically.

Explicit Expressions

$$Z_1(z, z^*, \mu) = \frac{1}{\pi} e^{2N\mu^2} \int_0^1 d\lambda \lambda e^{-2N\mu^2\lambda^2} I_0(\lambda z N) I_0(\lambda z^* N).$$

$$Z_{-1}(z, z^*, \mu) = \frac{N^3 \pi e^{-2N\mu^2}}{\mu^2} e^{-\frac{N(z^2-(z^*)^2)}{8\mu^2}} K_0\left(\frac{Nzz^*}{4\mu^2}\right).$$

Splittorff-JV-2003



$$\rho(r) = \frac{1}{2\pi} \int d\phi \rho(re^{i\phi}, re^{-i\phi}, \mu)$$

Comparison to numerical result for an ensemble of 2 million 200×200 matrices ($N = 200$). The asymptotic result is obtained from the large argument approximation of of K_0

Akemann-2003

III. Topology and Random Matrix Theory

Ten Fold Classification and Topology

Spectral Flow

Topology for non-hermitian RMT

Spectral Statistics and Topology

Topology and Random Matrix Theory

- Topology will be identified with the presence of zero modes.
- The GOE, GUE and GSE have no zero modes
- The ensemble of anti-symmetric Hermitian matrices (D) has topology. For odd-dimensions there is always a zero eigenvalue. Anti-selfdual Hermitian ensembles are even dimensional and do not have zero modes.
- The chiral ensemble CI has symmetric off-diagonal blocks and cannot have any zero modes.
- The chiral ensemble DIII has anti-symmetric off-diagonal blocks and can have one zero mode.
- The chiral ensembles, chGOE (BDI), chGUE (AII) and chGSE (CII), can have any integer number of zero modes.

Example: the chGUE

The chiral Gaussian Unitary Ensemble (chGUE or AIII) is the ensemble of Hermitian matrices with chiral symmetry and no anti-unitary symmetries JV-1994

$$H\Gamma_5^\nu + \Gamma_5^\nu H = 0, \qquad \Gamma_5^\nu = \mathrm{diag}(\underbrace{1, \cdots, 1}_{n}, \underbrace{-1, \cdots, -1}_{n+\nu}).$$

These matrices have the block form

$$H = \begin{pmatrix} 0 & C \\ C^\dagger & 0 \end{pmatrix}, \qquad C \text{ is } \; n \times (n+\nu) \text{ matrix}$$

$H(0,\psi)^T = 0$ gives $C\psi = 0$ which are n linear equation with $n+\nu$ unknowns, and we can find ν nontrivial solutions. Therefore, the Hamiltonian has ν zero modes. They are topological because they are insensitive to changing the nonzero matrix elements. The topology group is $\mathbb{Z}$.

Ten Fold Table

RMT	Chiral	Anti-Unitary	ZM
A (GUE)	No	None	No
AI (GOE)	No	$H^* = H$	No
AII (GSE)	No	$H^* = \Sigma_2 H \Sigma_2$	No
D$_\nu$	No	$H^* = -H$	$\mathbb{Z}_2$
C	No	$H^* = -\Sigma_2 H \Sigma_2$	No
AIII$_\nu$ (chGUE)	$\Gamma_5 H \Gamma_5 = -H$	None	$\mathbb{Z}$
BDI$_\nu$ (chGOE)	$\Gamma_5 H \Gamma_5 = -H$	$H^* = H$	$\mathbb{Z}$
CII$_\nu$ (chGSE)	$\Gamma_5 H \Gamma_5 = -H$	$\Sigma_2 H \Sigma_2 = H^*,$	$2\mathbb{Z}$
CI	$\Gamma_5 H \Gamma_5 = -H$	$H^* = -CHC^{-1}$	No
DIII$_\nu$	$\Gamma_5 H \Gamma_5 = -H$	$H^* = -\Sigma_2 H \Sigma_2$	$\mathbb{Z}_2$

Kieburg-JV-Zafeiropoulos-2014, Kieburg-Würfel-2017

The admissibility of topology is constrained by non-unitary symmetries.

García-García-Sá-JV-Yin-2023

Kitaev Versus Zero Modes

class	$d=0$	1	2	3
A	$\mathbb{Z}$		$\mathbb{Z}$	
AI	$\mathbb{Z}$			
AII	$2\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$
AIII		$\mathbb{Z}$		$\mathbb{Z}$
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$		
C			$2\mathbb{Z}$	
CI				$2\mathbb{Z}$
CII		$2\mathbb{Z}$		$\mathbb{Z}_2$
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	
DIII		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

Kitaev table of topological insulators Kitaev-2008, Schnyder-Ryu-Furusaki-Ludwig-2008

RMT	Chiral	Anti-Unitary	ZM
A (GUE)	No	None	No
AI (GOE)	No	$H^* = H$	No
AII (GSE)	No	$H^* = \Sigma_2 H \Sigma_2$	No
D_ν	No	$H^* = -H$	$\mathbb{Z}_2$
C	No	$H^* = -\Sigma_2 H \Sigma_2$	No
$AIII_\nu$ (chGUE)	$\Gamma_5 H \Gamma_5 = -H$	None	$\mathbb{Z}$
BDI_ν (chGOE)	$\Gamma_5 H \Gamma_5 = -H$	$H^* = H$	$\mathbb{Z}$
CII_ν (chGSE)	$\Gamma_5 H \Gamma_5 = -H$	$\Sigma_2 H \Sigma_2 = H^*,$	$2\mathbb{Z}$
CI	$\Gamma_5 H \Gamma_5 = -H$	$H^* = -CHC^{-1}$	No
$DIII_\nu$	$\Gamma_5 H \Gamma_5 = -H$	$H^* = -\Sigma_2 H \Sigma_2$	$\mathbb{Z}_2$

Example

Consider the matrix

$$D = \begin{pmatrix} m & 0 & ia \\ 0 & m & ib \\ ia^* & ib^* & m \end{pmatrix}, \qquad \Gamma_5^1 = \mathrm{diag}(1,1,-1)$$

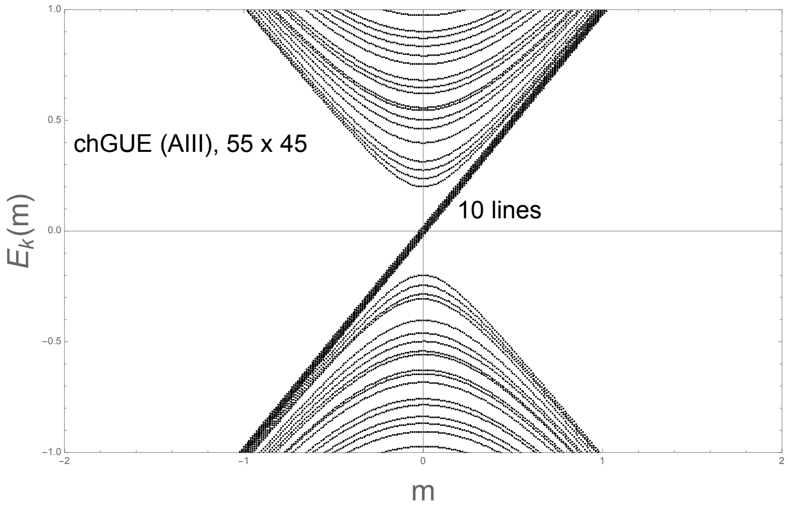
$$\Gamma_5^1 D = \begin{pmatrix} m & 0 & ia \\ 0 & m & ib \\ -ia^* & -ib^* & -m \end{pmatrix}.$$

The secular equation is

$$\det(D - \lambda \mathbf{1}) = (m-\lambda)(\lambda^2 - m^2 - b^* b) - (m-\lambda) a^* a = 0.$$

This gives the eigenvalues: $\lambda = m, \lambda = \pm\sqrt{m^2 + a^* a + b^* b}$.

Spectral Flow and Topology



Because of level repulsion, the nonzero eigenvalues are shifted by $(\nu/2)\Delta\lambda$.

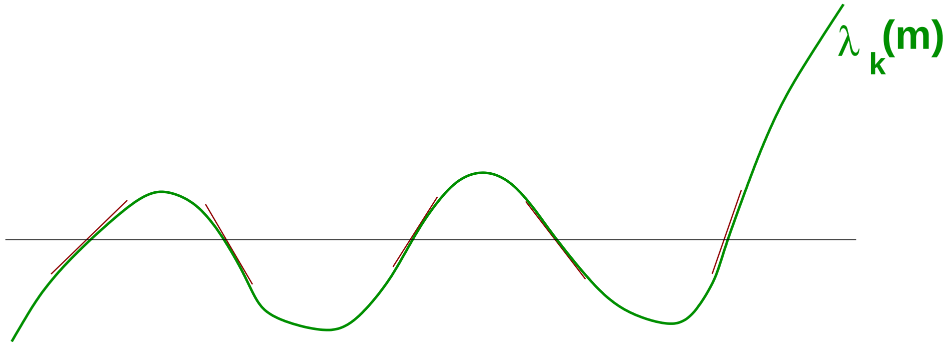
Spectral flow of

$$\Gamma_5^\nu \begin{pmatrix} m & id \\ id^\dagger & m \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} m & id \\ id^\dagger & m \end{pmatrix} = \begin{pmatrix} m & id \\ -id^\dagger & -m \end{pmatrix}$$

.

Index

- The mode corresponding to the zero eigenvalue crosses the m-axis, but the other modes do not.
- When we deform the $\lambda(m) = m$ line, it may cross the m-axis multiple times. The additional crossings come in pairs, one with a positive derivative and the other one with a negative derivative.



Topological Index

A topological index can be defined by

$$\nu = \sum_k \int_{-\infty}^{\infty} dm \delta(\lambda_k(m)) \frac{d\lambda_k(m)}{dm} \quad = \quad \sum_k \int_{-\infty}^{\infty} dm \frac{1}{2} \frac{d}{dm} \mathrm{sign} \lambda_k(m).$$

A more conventional definition of the index is obtained for

$$D = \begin{pmatrix} m^{-i\phi} & id \\ id^\dagger & me^{i\phi} \end{pmatrix}.$$

Then, $\det D = e^{i\nu\phi} F(|m|)$, and

$$\begin{aligned} \nu &= \frac{1}{2\pi} \int d\phi \frac{d}{d\phi} \sum_k \arg(\lambda_k) = \frac{1}{2\pi} \int d\phi \frac{d}{d\phi} \arg(\det D) \\ &= \frac{1}{2\pi i} \int d\phi \frac{d}{d\phi} \log(\det D). \end{aligned}$$

Itoh-Iwasaki-Yoshie-1986, Kawabata-Shiozaki-Ryu-2022

Topology for non-Hermitian RMTs

- For Hermitian matrices, topology is related to zero modes which can result from anti-symmetric or rectangular blocks
- Non-Hermitian random matrices can be characterized in terms of 1×1, 2×2 or 4×4 block matrices each with or without anti-unitary symmetries
- For non-Hermitian systems we generalize the concept of topology to the presence of anti-symmetric or rectangular blocks
- It is not yet known which of the 38 non-Hermitian random matrix ensembles have topological extensions

Example: The Ensemble AIII[†]

This ensemble has the block structure

$$H = \begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix}$$

with C and D complex matrices. We can choose C and D to be $n \times (n + \nu)$ and $(n + \nu) \times n$ complex rectangular matrices. Then H will have exactly ν zero modes.

The joint eigenvalue distribution of this model is known even if C and D do not have the same width James Osborn - 2004 . The joint eigenvalue distribution for $\beta = 4$ is also known Akemann-Bittner-2006.

This ensemble has been studied extensively as a model for QCD at nonzero chemical potential, see for example Akemann-Osborn-Splittorff-JV-2005.

Example: The Ensemble AIII

This random matrix Ensemble has the block structure

$$H = \begin{pmatrix} aA & C \\ -C^\dagger & aB \end{pmatrix}$$

with A and B Hermitian and C a complex iid Gaussian random variables. The C block can be chosen to be an $n \times (n+\nu)$ matrix and. This ensemble is pseudo-Hermitian,

$$(\Gamma_5^\nu H)^\dagger = \Gamma_5^\nu H.$$

It was first introduced for the study of the Wilson Dirac operator. Damgaard-Splittorff-JV-2010, Akemann-Damgaard-Splittorff-JV-2011. The joint eigenvalue distribution of both H (Kieburg-JV-Zafeiropoulos-2011) and $\Gamma_5^\nu H$ are known (Akemann-Nagao-2011).

Spectral Flow

For this RMT we can consider the spectral flow of $\Gamma_5^\nu(H+m)$ as a function of real m. If

$$\Gamma_5^\nu(H+m_0)\phi = 0$$

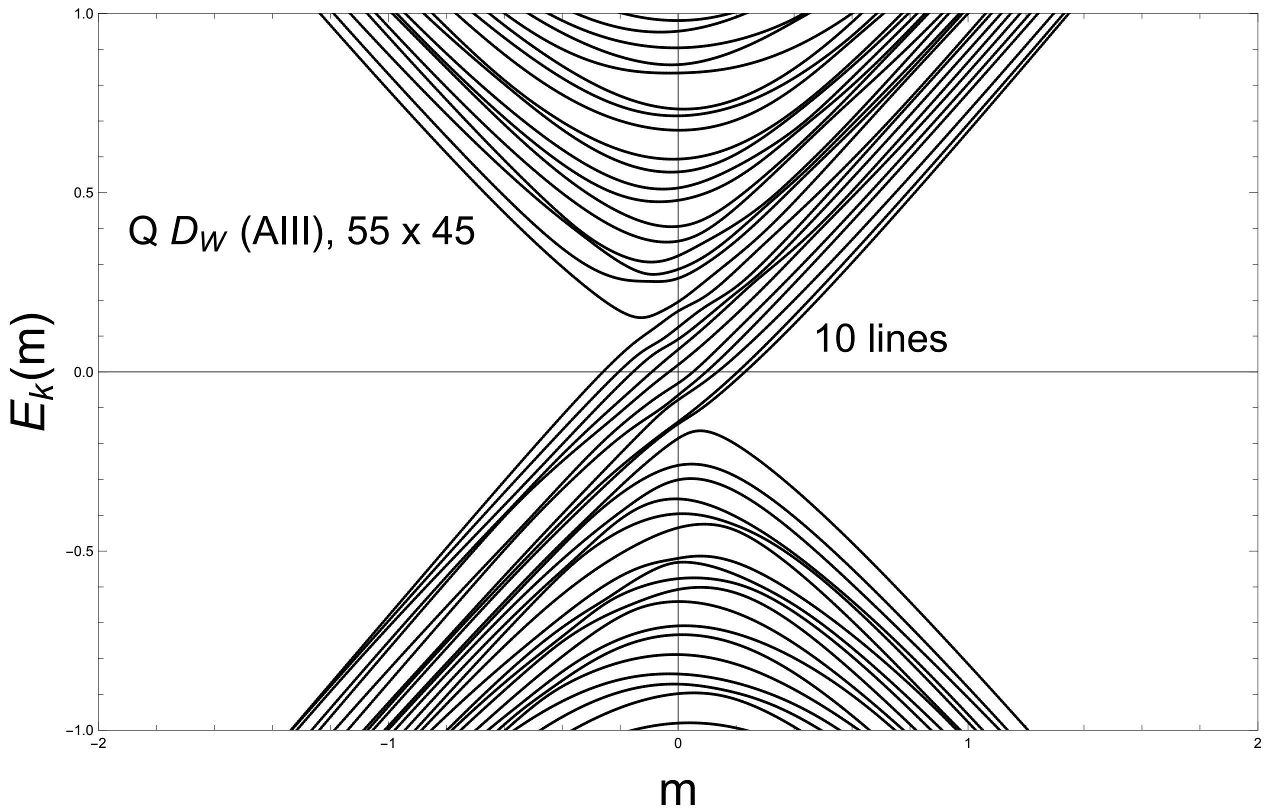
for some value of m_0, then

$$H\phi = -m_0\phi.$$

Therefore, real eigenvalues of H correspond to intersection of the spectral flow lines of $\Gamma_5^\nu(H+m)$.

We can also consider the spectral flow as a function of the non-Hermiticity parameter a,

Spectral Flow of Real Eigenvalues of AIII$_\nu$

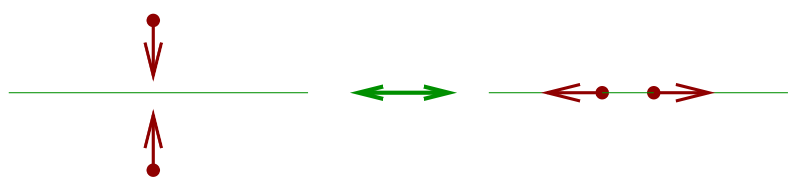


$$QD \equiv \Gamma_5^{10} D = \begin{pmatrix} aA + m & id \\ -id^\dagger & -aB - m \end{pmatrix}, \qquad A^\dagger = A, B^\dagger = B.$$

Akemann-Damgaard-Splittorff-JV-2011, Kieburg-JV-Zafeiropoulos-2012

Two kinds of real eigenvalues

- Eigenvalues that are connected to spectral flow lines that intersect the m-axis.
- Real eigenvalues that arise from the collision of a pair of complex conjugate eigenvalues Note that intersections of flow lines with the $m-$axis cannot disappear. So two topological real eigenvalues cannot collide to become a pair of complex conjugate real eigenvalues.

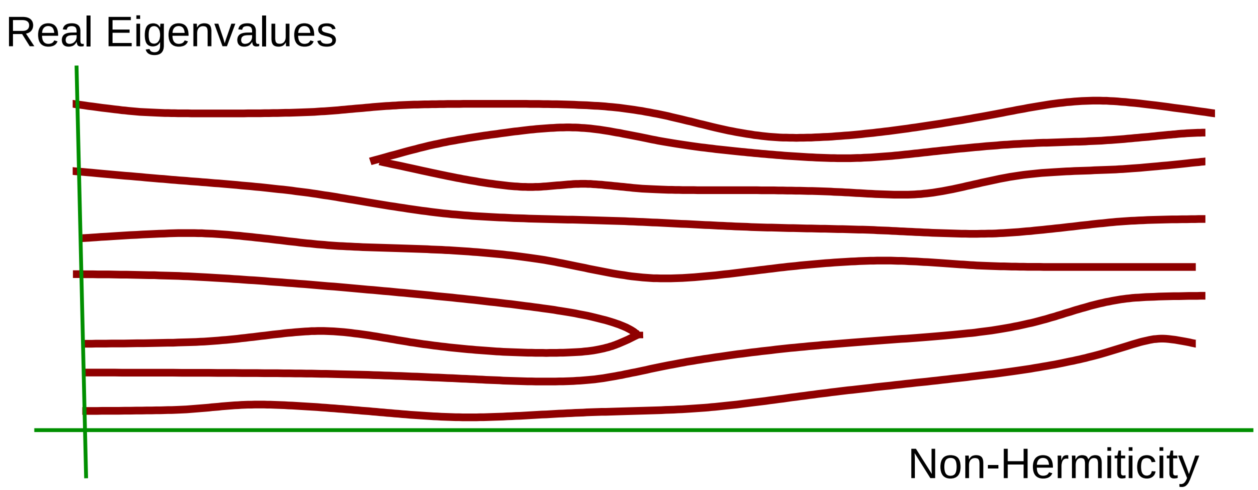


The spectral statistics of the two types of real eigenvalues is very different.

Spectral Statistics of Non-Topological Real Eigenvalues

- From Gernot's talk we learned that there are only 5 bulk statistics classes for the 38 non-Hermitian random matrix ensembles.
- This raises the question whether we can find other spectral statistics to distinguish the ensembles
- We already have seen that ensembles with chiral symmetry, can be characterized by the microscopic spectral density
- Another idea is to look at the distribution of the real eigenvalues.
- It should be clear that topological real eigenvalues have a distribution that is completely different from real eigenvalues originating from a complex conjugate pair.

Spectral Statistics of Non-Topological Real Eigenvalues



We find close eigenvalues and large gaps because of two real eigenvalues that become a complex conjugate pair and two complex eigenvalues that join the real axis. This results in characteristic correlations of the real eigenvalues which are not given by the Hermitian random matrix ensembles.

Spectral correlation of topological eigenvalues are given by Hermitian RMTs. García-García-Sá-JV-Yin-2023, 2024

Zero Modes and Real Eigenvalues

Intersections of the flow lines with the m-axes

$$Q(D_W + m_k)\phi_k = 0.$$

Then

$$D_W \phi_k = -m_k \phi_k.$$

An intersection of a flow line with the m-axis corresponds to a real eigenvalue of D_W. The parity of the intersection number of a flow line is conserved under continuous deformations of the Hamiltonian, e.g. by changing the non-Hermiticity parameter. So the total number of flow line that go from bottom-left to top-right is a topological invariant.

IV. The Non-Hermitian Sachdev-Ye-Kitaev Model

Model

Spectral Flow

Gap Ratio Distributions

Stability of Topological Modes

The Non-Hermitian Sachdev-Ye-Kitaev Model

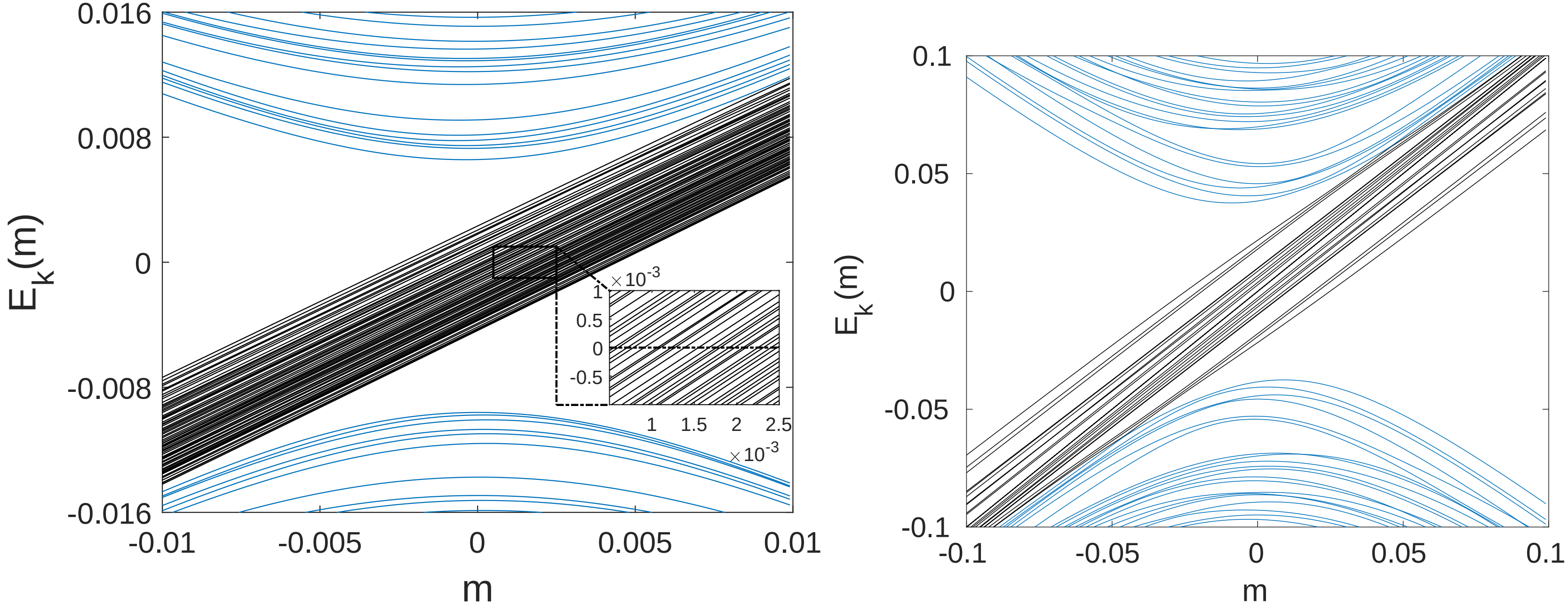
This model is defined by the Hamiltonian

$$H = \sum (iM_{ijkl}\psi_i^L\psi_j^L\psi_k^L\psi_l^L - iM_{ijkl}\psi_i^R\psi_j^R\psi_k^R\psi_l^R) + \lambda \left(i\sum_k \psi_k^L\psi_k^R \right)^r .$$

Maldacena-Qi-2018, García-García-Nosaka-Rosa-JV-2019, García-García-Jia-Rosa-JV-2021, García-García-Sá-JV-Yin-2024

This model shows emergent topology. Depending on the quantum numbers block of the Hamiltonian may be rectangular resulting in zero modes for $\lambda = 0$. the number of zero modes is on the order of the square root of the total size of the matrix, i.e. $O(2^{N/4})$.

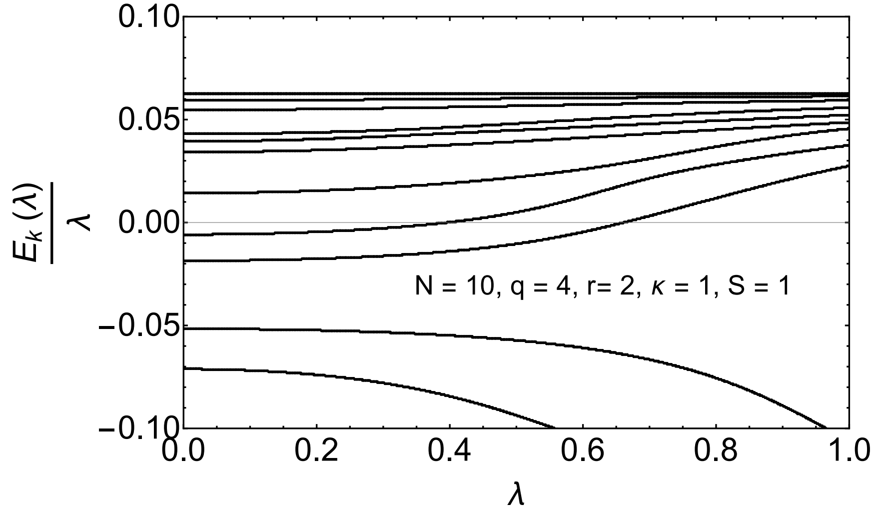
Spectral flow of Lindbladian SYK

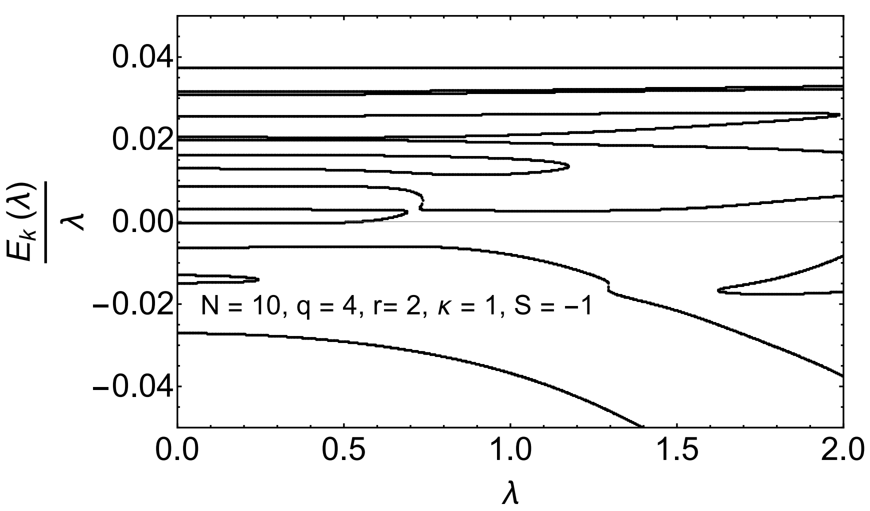


Left: Spectral flow for $N = 14,\ q = 4,\ r = 2,\ \lambda = 0.1,\ S_L = S_R = -1$ (AIII$_\nu$) . Right: Spectral flow for $N = 10,\ q = 4,\ r = 1,\ \lambda = 0.02,\ S = 1$ (CI$_{--\nu}$).

Note the significant gap due to the level repulsion of the eigenvalues at zero.

Flow of Real Eigenvalues with Non-Hermiticity

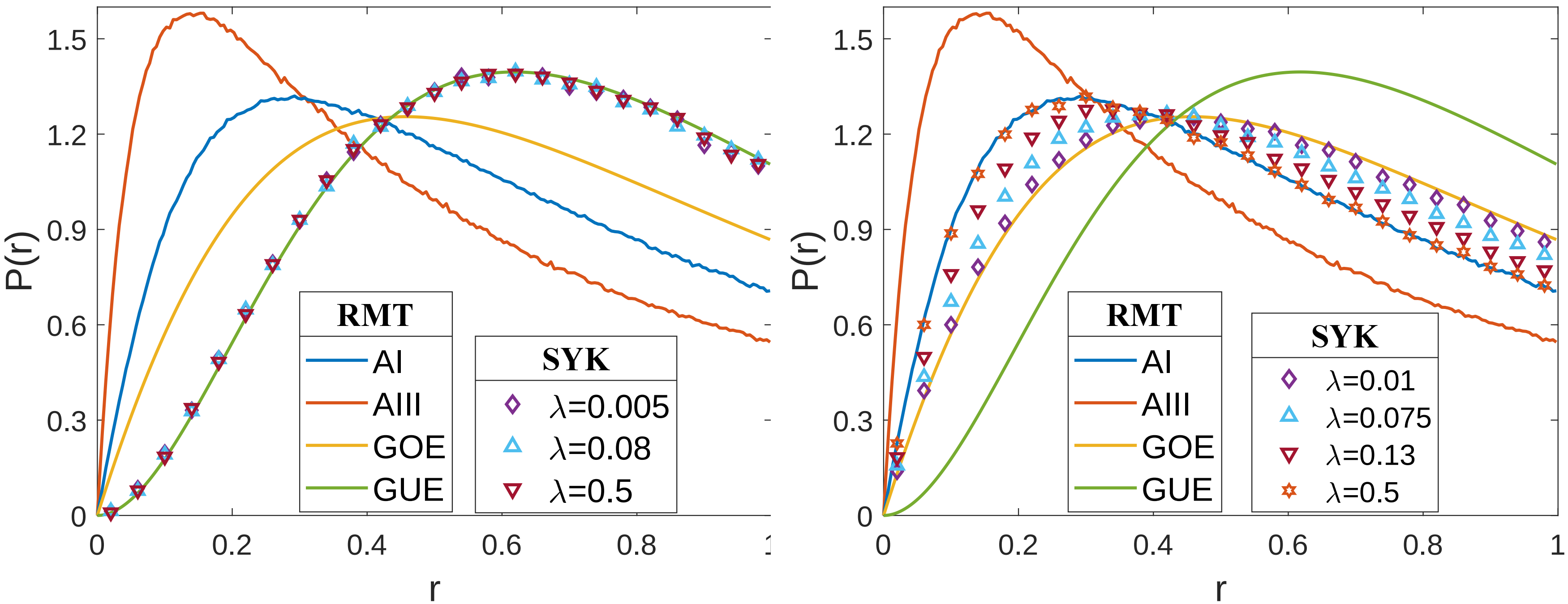




Spectral flow of the real eigenvalues of $\mathcal{L})$ as a function of the LR interaction parameter λ for $N = 10$, $q = 4$ and $S = 1$ (left, AIII$_\nu$) and $S = -1$ (right, AI). The eigenvalues in the left figure are topological while in the right figure there are no topological real eigenvalues.

García-García-Sá-JV-Yin-2024

Gap Ratio Distribution

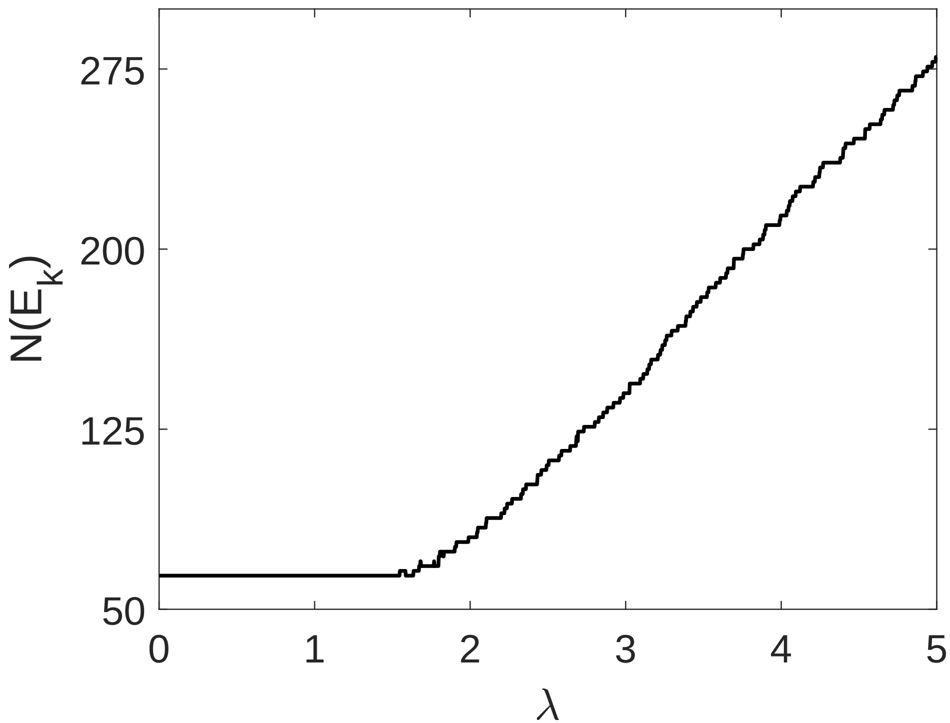


Gap ratio distribution for the same universality class as in previous slide.

Note that two different blocks of the same Hamiltonian have different level statistics, This can happen because the square of an anti-unitary symmetry depends on the quantum numbers $(QS_L K)^2 = (-1)^{N/2} S$.

García-García-Sá-JV-Yin-2023

Stability of Topological Modes



Number of real eigenvalues versus non-Hermiticity parameter λ for $N = 12$, $q = 4$, $r = 2$ and $S_L = S_R = 1$ (class $\mathrm{BDI}^{\dagger}_{\nu}$).

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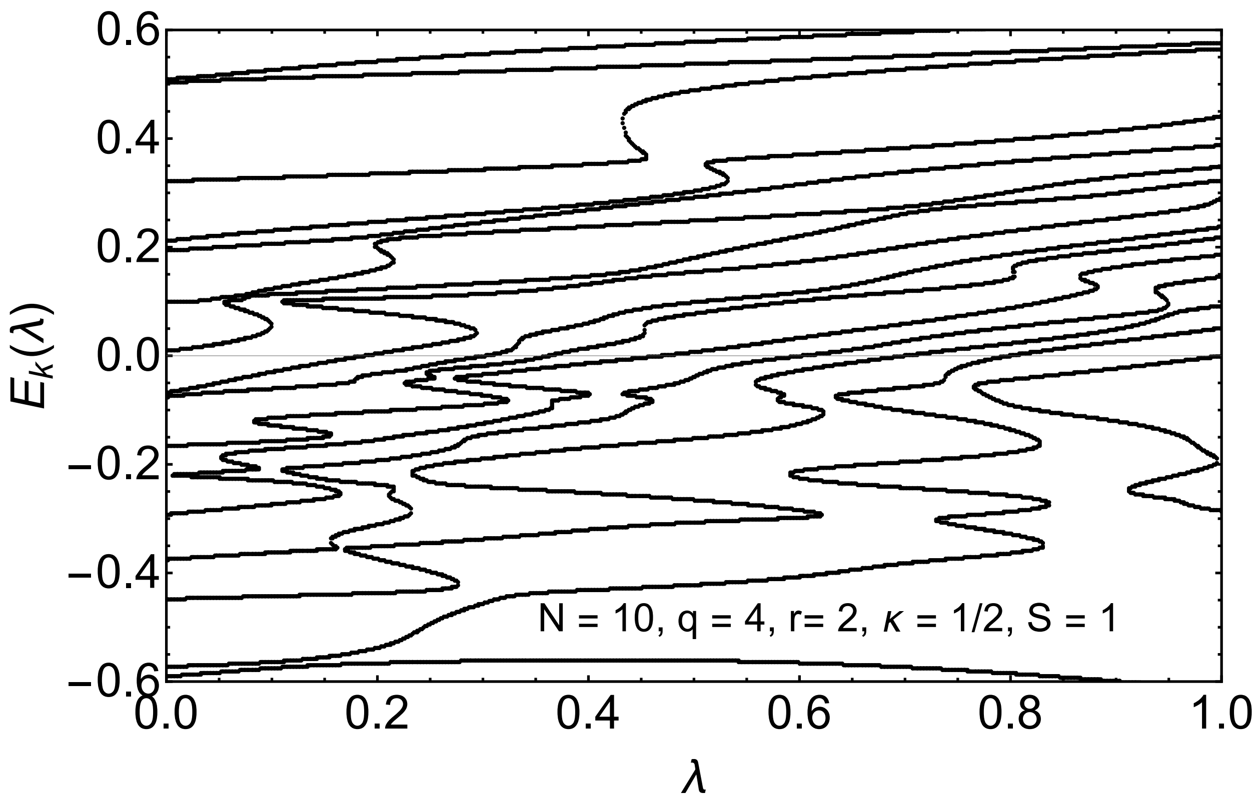
Spectral Flow in Absence of Topology



Spectral flow in absence of topology for $N = 10$, $q = 4$, $r = 2$, $\kappa = 1/2$ and $S = 1$ (class A).

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V. Conclusions

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- Integrability is a powerfui method to obtain the spectral density of $\beta = 2$ random matrix theories.
- Anti-unitary symmetries also determine the topological properties on non-Hermitian random matrix ensembles.
- We can distinguish topological real eigenvalues and real eigenvalues that result from a pair of complex conjugate eigenvalues joining the real axis.
- A gap $\sim \nu$ is induced by level repulsion guarantees the stability of the the topological modes

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Classification Summary

N	$(C_1K)^2$	$(C_2K)^2$	C_1KC_2K	RMT	Matrix Elements
2	1	-1	$-i\Gamma_5$	GUE	Complex
4	-1	-1	$-\Gamma_5$	GSE	Quaternion
6	-1	1	$-i\Gamma_5$	GUE	Complex
8	1	1	Γ_5	GOE	Real
10	1	-1	$-i\Gamma_5$	GUE	Complex
12	-1	-1	Γ_5	GSE	Quaternion

Table 1: (Anti-)Unitary symmetries of the SYK Hamiltonian and the corresponding RMT. The symmetries are periodic in N modulo 8 (Bott periodicity). You-Ludwig-Xu-2016, Garcia-Garcia-JV-2016

A similar classification exists for supersymmetric SYK models.
Fu-Gaiotto-Maldacena-Sachdev-2017, Li-Liu-Xin-Zhou-2017, Kanazawa-Wettig-2017

Upper Bound for Lyapunov Exponent

Lyapunov exponent λ

$$\Delta(t) \sim \Delta(0)e^{\lambda t}$$

Energy-time "uncertainty relation"

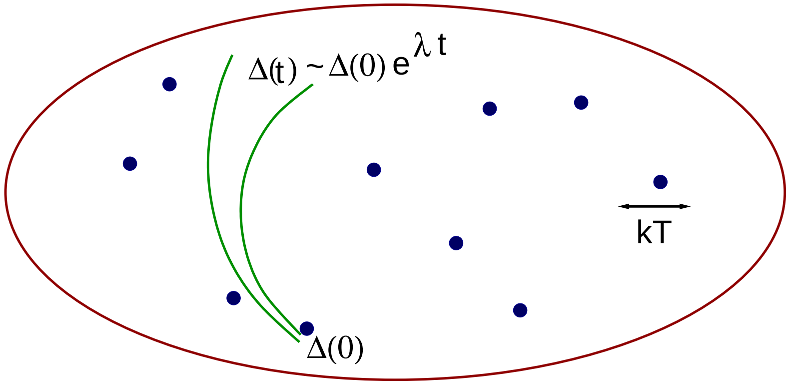
$$\begin{aligned} \Delta t \Delta E &\geq \frac{\hbar}{2} \\ \Delta t &\sim 1/\lambda, \qquad \Delta E \sim \pi k T \end{aligned}$$

So we have the bound

$$\lambda \leq \frac{2\pi k T}{\hbar}$$

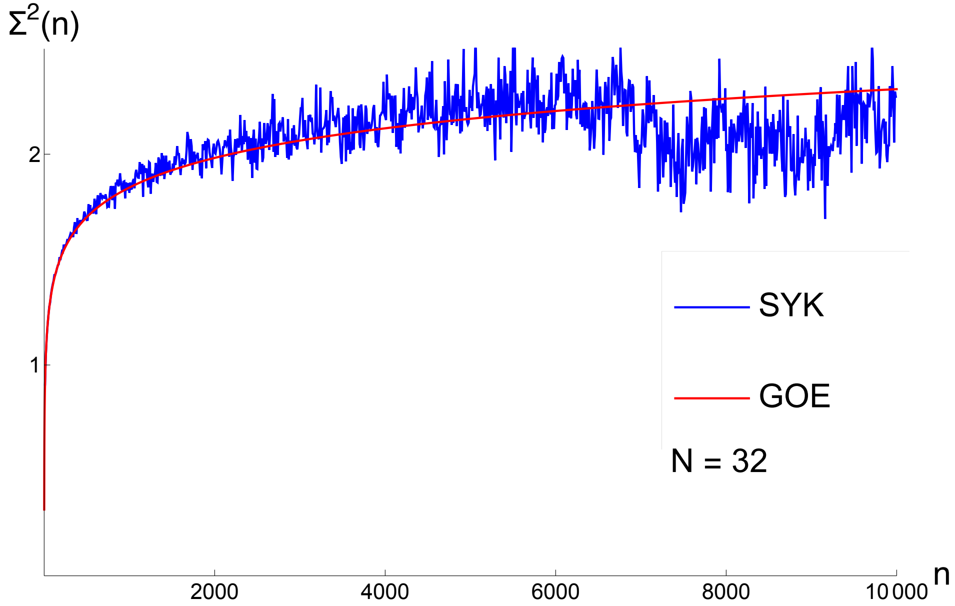
Maldacena-Shenker-Stanford-2015

Of the same type as the η/S bound by Son.



Divergence of trajectories in a stadium at temperature T

Maximum Range of Agreement with RMT



Number variance after unfolding configuration by configuration. The total number of eigenvalues for $N = 32$ is 32,768.