Linear statistics for Coulomb and Riesz gases: higher order cumulants

Grégory Schehr
Laboratoire de Physique Théorique et Hautes Energies,
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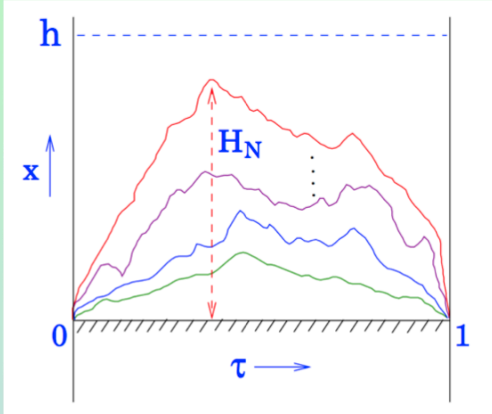
Log-gases in Caeli Australi
MATRIX Institute, Creswick, 4-15 August 2025

An exact expression for $F_N(h)$

- Distribution of the maximal height $\mathcal{H}_N$

$$F_N(h) \quad = \quad \mathbb{P}\left[x_N(\tau) \leq h \,,\, \forall\, 0 \leq \tau \leq 1\right]$$



- Exact result for finite N G. S., S. N. Majumdar, A. Comtet, J. Randon-Furling '08

$$F_N(h) \quad = \quad \frac{A_N}{h^{2N^2+N}} \sum_{n_1,\cdots,n_N=0}^{+\infty} \prod_{i=1}^{N} n_i^2 \prod_{1\leq j<k\leq N} (n_j^2 - n_k^2)^2 e^{-\frac{\pi^2}{2h^2}\sum_{i=1}^N n_i^2}$$

$$A_N = \frac{\pi^{2N^2+N}}{2^{N^2+\frac{N}{2}} \prod_{j=0}^{N-1} \Gamma(2+j)\Gamma(\frac{3}{2}+j)}$$

see also T. Feierl, M. Katori *et al.* '08

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in collaboration with

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A. Flack, S. N. Majumdar, G. S., J. Phys. A **56**, 105002 (2023)

B. De Bruyne, P. Le Doussal, S. N. Majumdar, G. S., J. Phys. A **57**, 155002 (2024)

P. Le Doussal, G. S., J. Stat. Phys. **192**, 1 (2025)

Outline

- Motivations and background on linear statistics in RMT/Coulomb gases
- Coulomb gases in d-dimensions: main results
- One-dimensional Riesz gases: main results
- Sketch of the derivation
- Conclusion and perspectives

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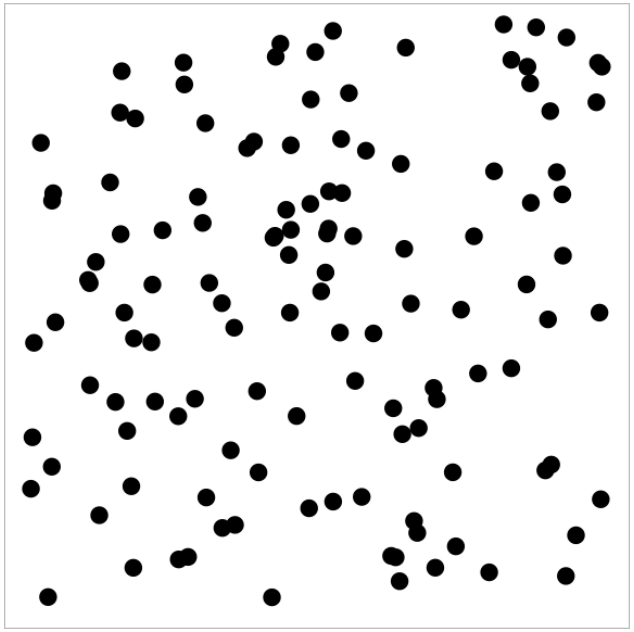
Linear statistics of random point processes



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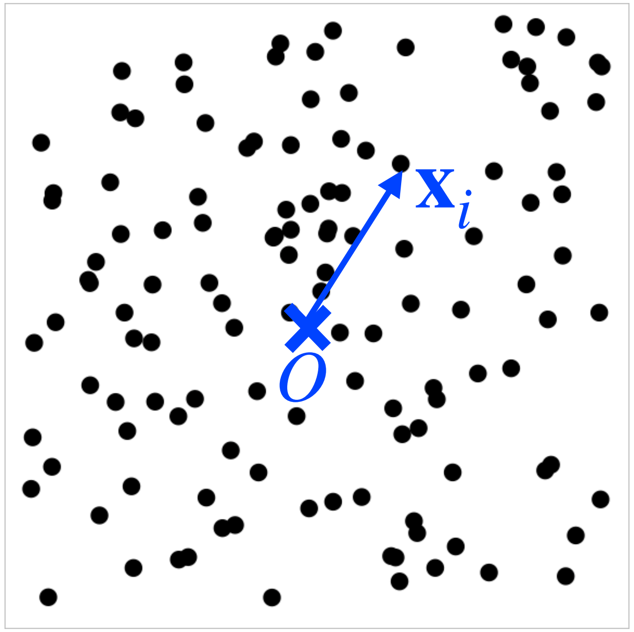

- Let us consider a gas of N particles, with positions $\mathbf{x}_i = \{x_i^{(1)}, x_i^{(2)}, \cdots, x_i^{(d)}\} \in \mathbb{R}^d$, with $i = 1,2,\cdots,N$

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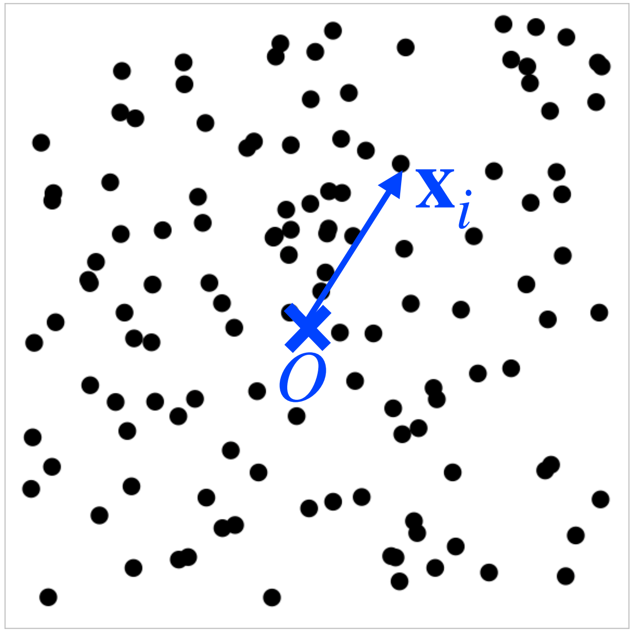

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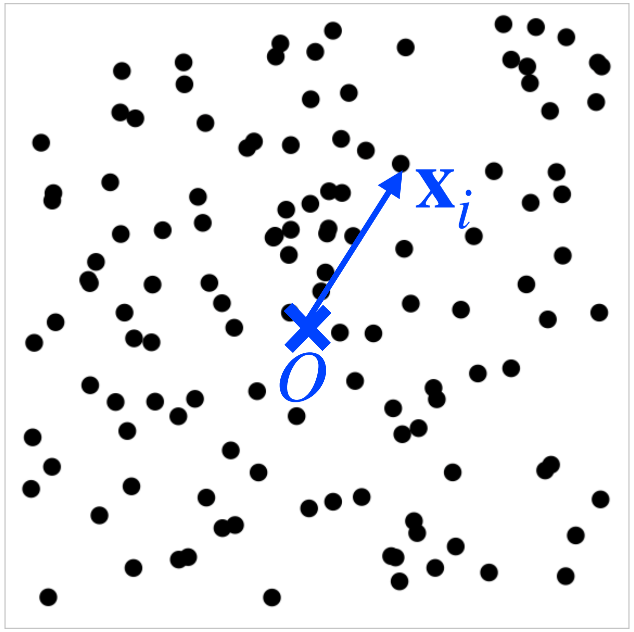

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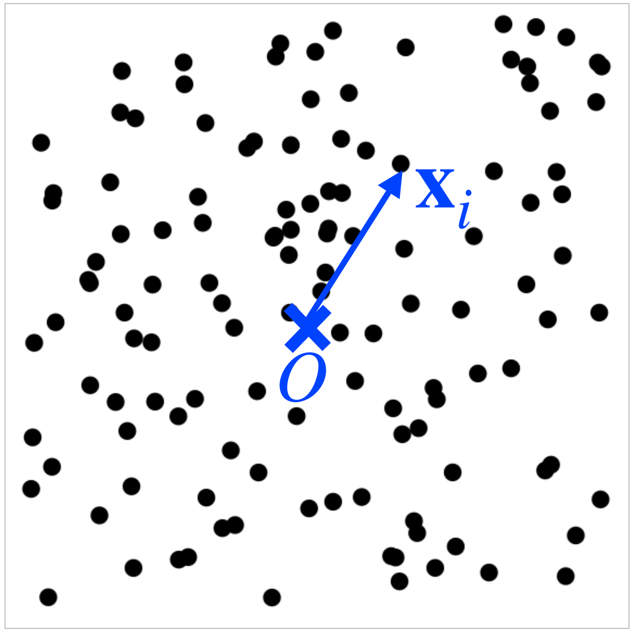

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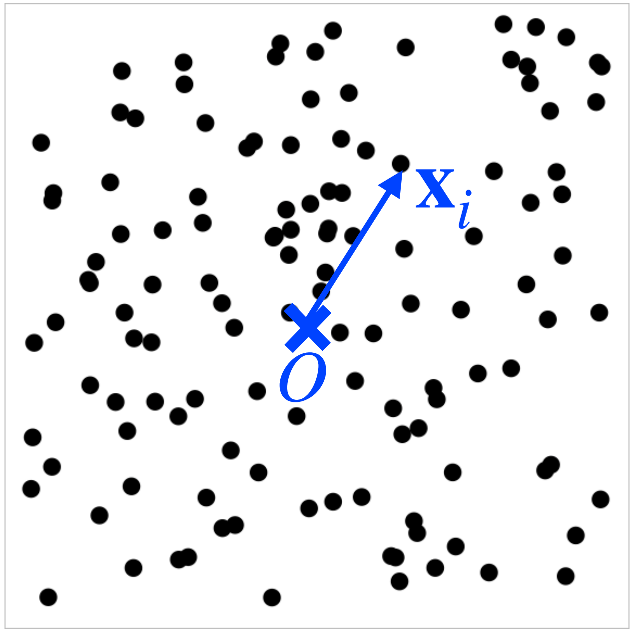

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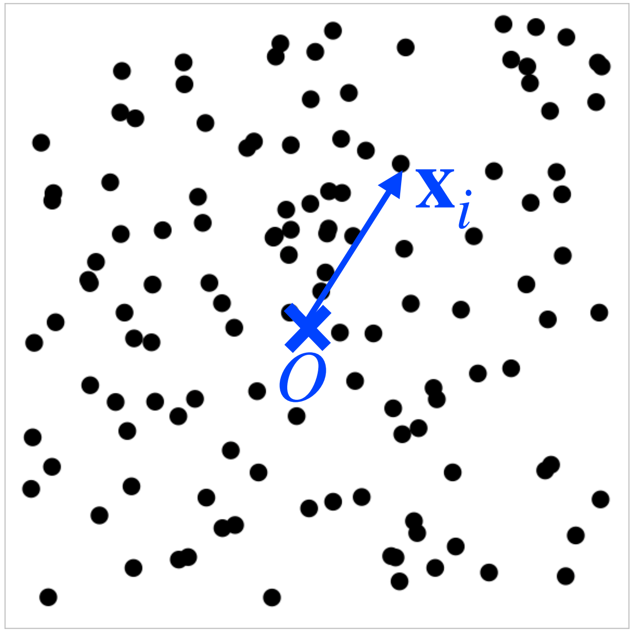

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What about correlated point processes ?

E.g. random matrices and their cousins (Coulomb/Riesz gas)

Linear statistics for Gaussian β-ensembles

$$P(\lambda_1, \cdots, \lambda_N) = \frac{1}{Z_N} \prod_{i<j} |\lambda_i - \lambda_j|^\beta e^{-\frac{\beta N}{2} \sum_{i=1}^N \lambda_i^2}$$

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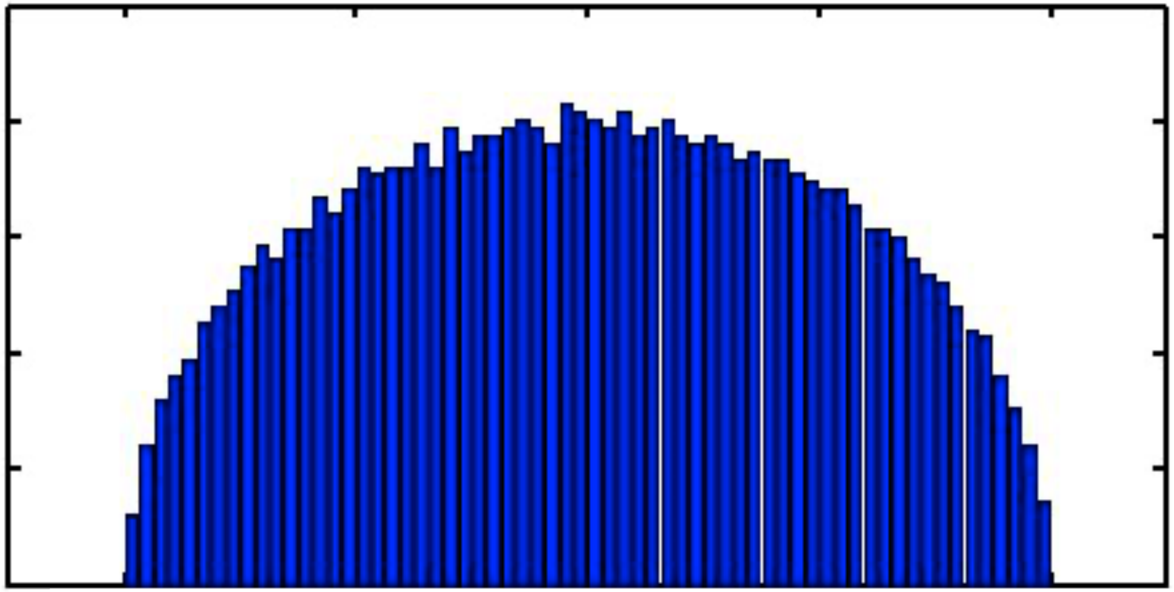

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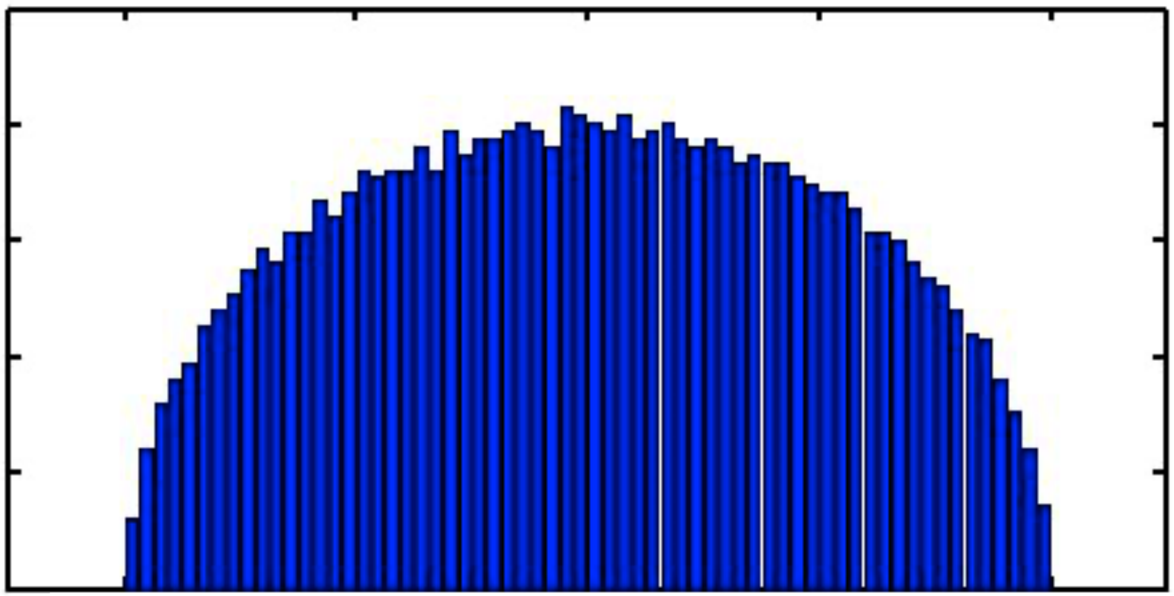

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  - Examples: $f(x) = x^n$ (Rényi entropy), $f(x) = x(1-x)$ (chaotic transport), etc... Beenakker, Bohigas, Damle, Kanzieper, Majumdar, Nadal, Savin, Sommers, Texier, Vergassola, Vivo

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  - Disclaimer: the rest of this talk excludes the case of an indicator function $f(x) = I_J(x)$ – full counting statistics – and focuses on smooth functions $f(x)$

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Brézin-Zee '93, Beenakker '93, Johansson '98, Scherbina '13, Bourgade-Erdös-Yau '14, Cunden-Mezzadri-Vivo '15, Beckerman-Leblé-Serfaty '17, Lambert-Ledoux-Webb '17, …

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This talk: what about higher cumulants for one-dimensional log-gases (and their cousins the Riesz gases) ?

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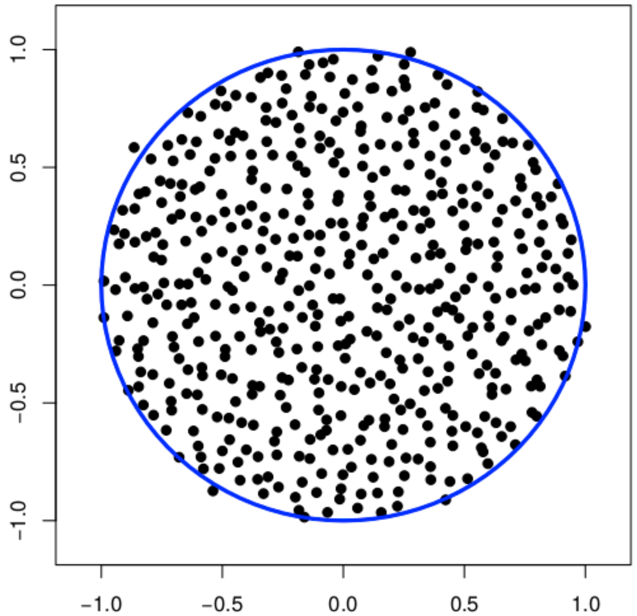

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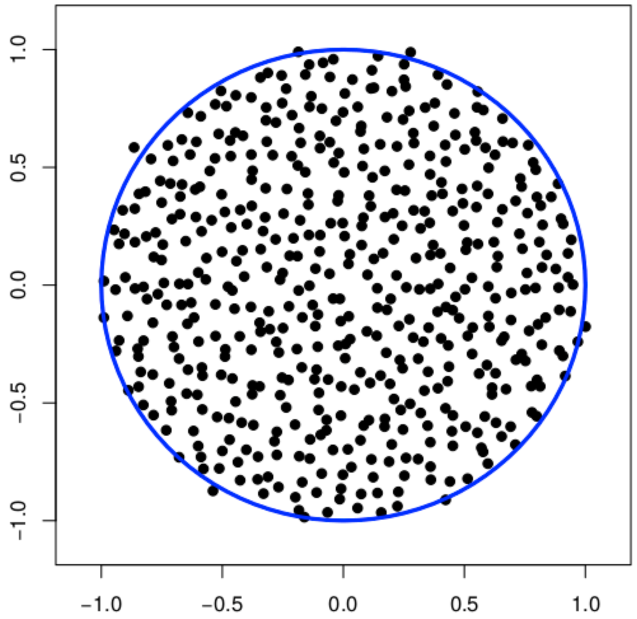

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where $\Sigma^2(f) = \sigma_f^2 + \tilde{\sigma}_f^2$

Linear statistics for complex Ginibre ensemble

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where $\hat{f}(k) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{-ik\theta} \, d\theta$

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$$\boxed{\Sigma^2(f)=\frac{1}{4\pi}\int_{\mathbb{C}}\left(\nabla f^H\right)^2 d^2z}$$

where f^H is the harmonic extension of f outside $\mathbb{D}(0,1)$, i.e., f^H is continuous, with $f^H(z)=f(z)$ for $z\in\mathbb{D}(0,1)$ and $\Delta f^H=0$ for $z\in\overline{\mathbb{D}(0,1)}$

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- For rotationally invariant linear statistics $f(z) = f(|z|)$,

$\tilde{\sigma}_f = 0$ (no boundary term) Forrester '99

Linear statistics for normal matrices

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- Equilibrium density

$$\rho_N(z) = \frac{1}{N} \sum_{i=1}^{N} \delta^{(2)}(z - z_i) \xrightarrow[N\to\infty]{} \rho_{\rm eq}(z) = \frac{1}{4\pi} \Delta V \quad , \quad z \in \mathbb{D}(0,R)$$

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Ameur, Hedenmalm & Makarov '11

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Linear statistics for general 2d Coulomb gases

$$P(z_1, \cdots, z_N) = \frac{1}{Z_N} \prod_{i<j} |z_i - z_j|^\beta e^{-\beta N \sum_{i=1}^{N} V(|z_i)|}$$

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  - Flack, Majumdar, G. S. '22 for d=1

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$\boxed{\text{What about higher cumulants for d-dimensional Coulomb gas ?}}$

Outline

- Motivations and background on linear statistics in RMT/Coulomb gases
- Coulomb gases in d-dimensions: main results
- One-dimensional Riesz gases: main results
- Sketch of the derivation
- Conclusion and perspectives

Linear statistics for the d-dimensional Coulomb gas

$$P(\mathbf{x}_1, \ldots, \mathbf{x}_N) = \frac{1}{Z_{N,V}} e^{-\beta E(\mathbf{x}_1, \ldots, \mathbf{x}_N)}$$

where $$E(\mathbf{x}_1, \ldots, \mathbf{x}_N) = \sum_{1 \le i < j \le N} U_d(|\mathbf{x}_i - \mathbf{x}_j|) + N \sum_{k=1}^{N} V(|\mathbf{x}_k|)$$

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Coulomb interaction $$U_d(x) = \begin{cases} \frac{1}{d-2} x^{d-2} & d \neq 2 \\ -\log(x) & d = 2 \end{cases}$$

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Q: Cumulants of $\mathscr{L}_N = \sum_{i=1}^{N} f(|\mathbf{x}_i|)$?

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$$\rho_{\rm eq}(x = |\mathbf{x}|) = \frac{1}{\Omega_d} \Delta V(|\mathbf{x}|) = \frac{1}{\Omega_d} \frac{1}{x^{d-1}} \left(x^{d-1} V'(x)\right)' \quad , \quad 0 \le x \le R$$

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Results for the higher cumulants of $\mathscr{L}_N$

De Bruyne, Le Doussal, Majumdar, G. S. '23

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- For example

$$\langle \mathscr{L}_N^3 \rangle_c \sim \frac{1}{\beta^2 N} \frac{R^{d-1} f'(R)^3}{V''(R) + \frac{d-1}{R^d}} \quad , \quad V'(R) R^{d-1} = 1$$

$$\langle \mathscr{L}_N^4 \rangle_c \sim \frac{1}{\beta^3 N^2} \left(\frac{\left((d-1)d - R^{d+1} V^{(3)}(R)\right) f'(R)^4}{R^2 \left(V''(R) + \frac{d-1}{R^d}\right)^3} + \frac{R^{d-2} f'(R)^3 \left((d-1) f'(R) + 4 R f''(R)\right)}{\left(V''(R) + \frac{d-1}{R^d}\right)^2} \right)$$

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- Some cases of (V, f) can be worked out explicitly to any order q

Outline

- Motivations and background on linear statistics in RMT/Coulomb gases
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Linear statistics for the 1d Riesz gas

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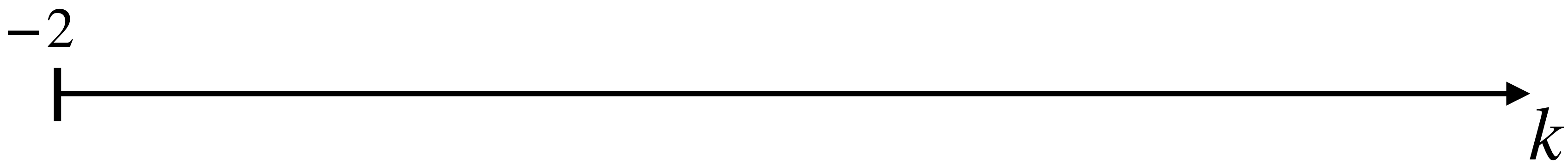

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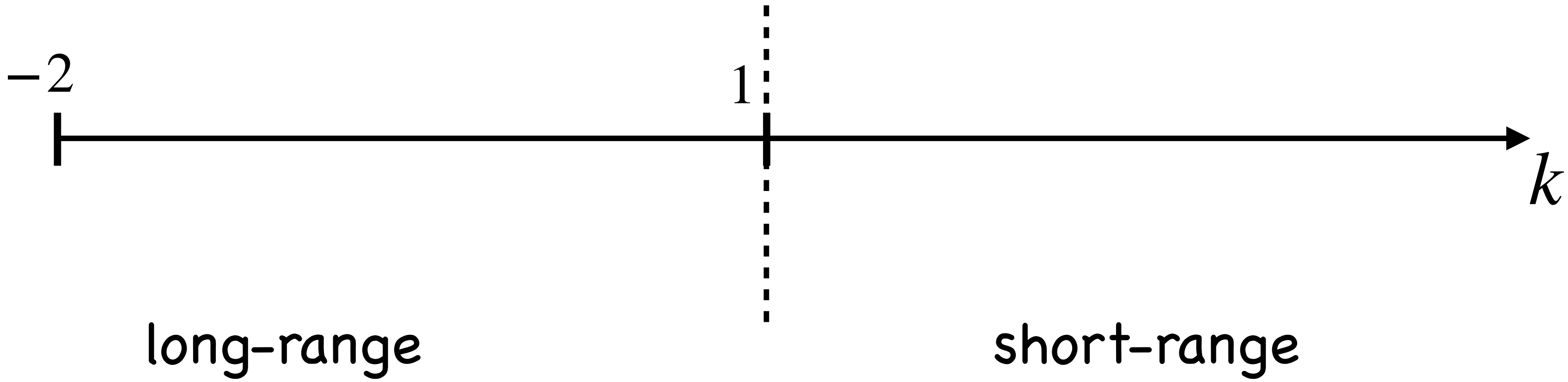

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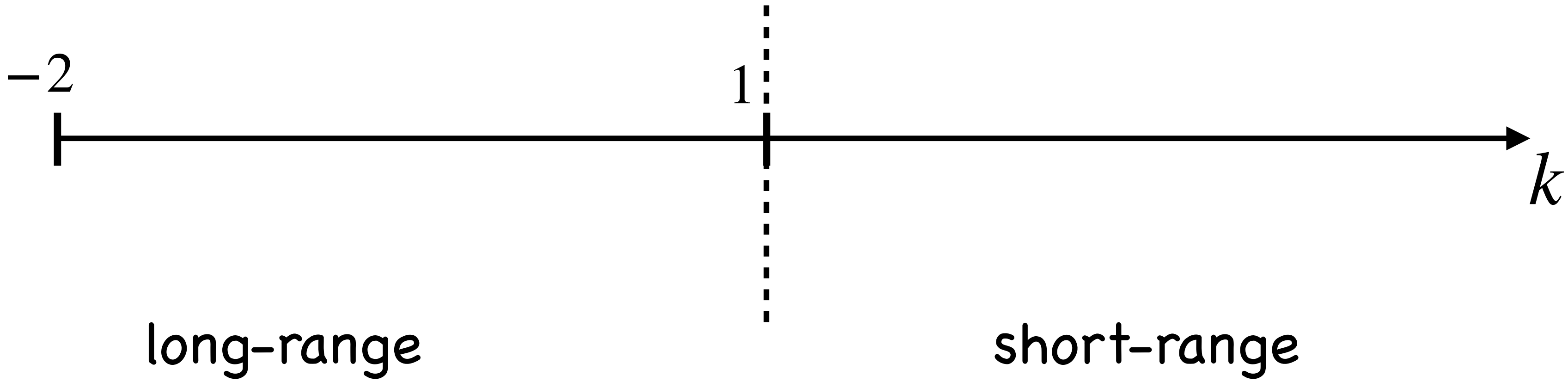

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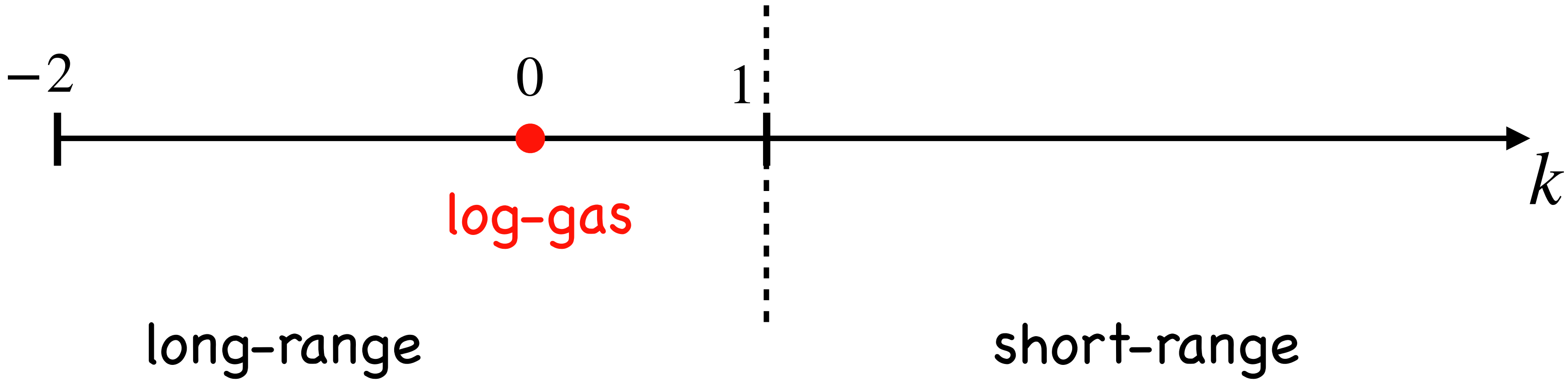

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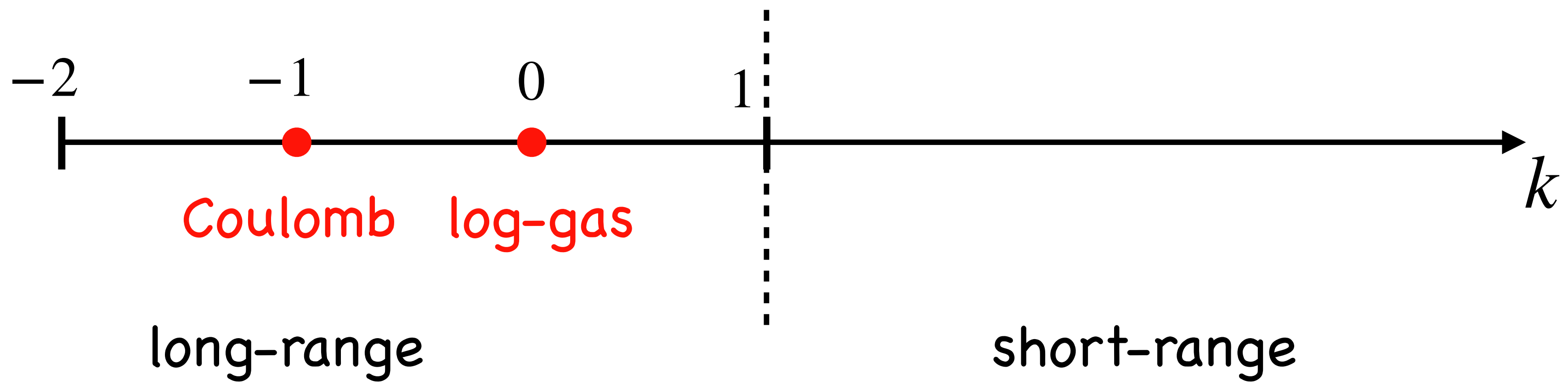

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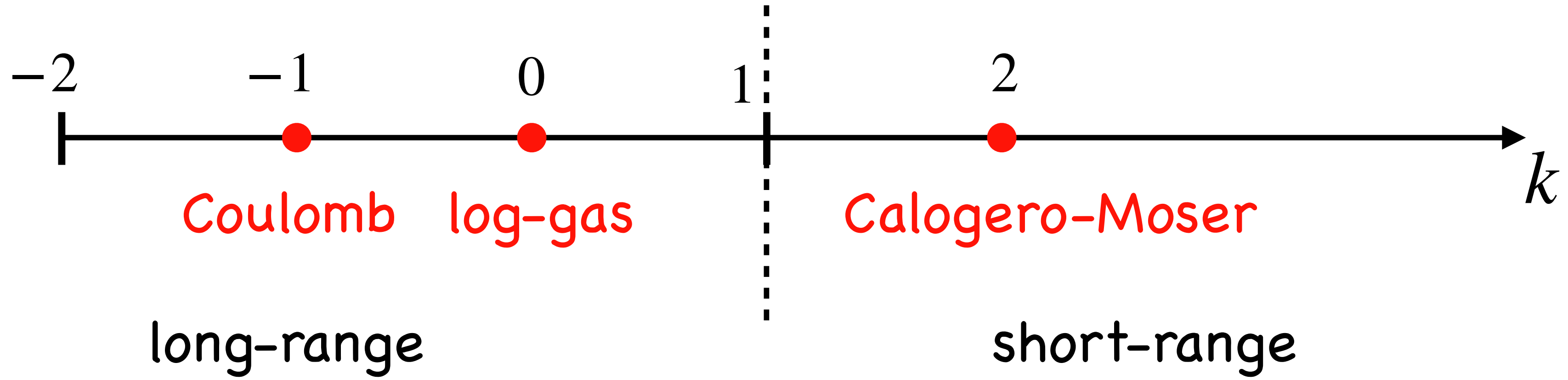

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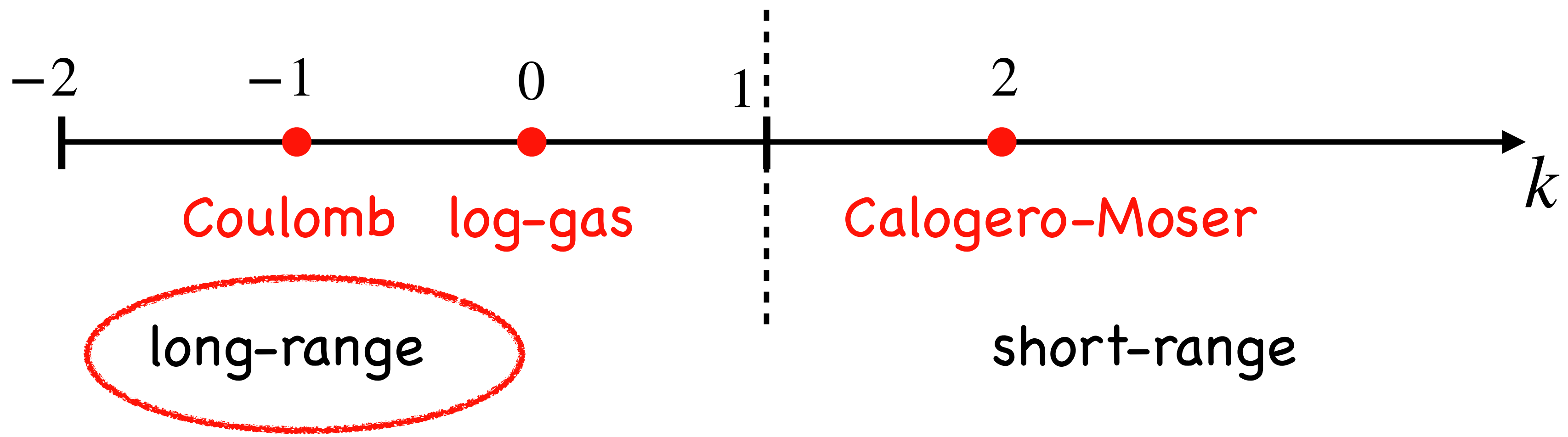

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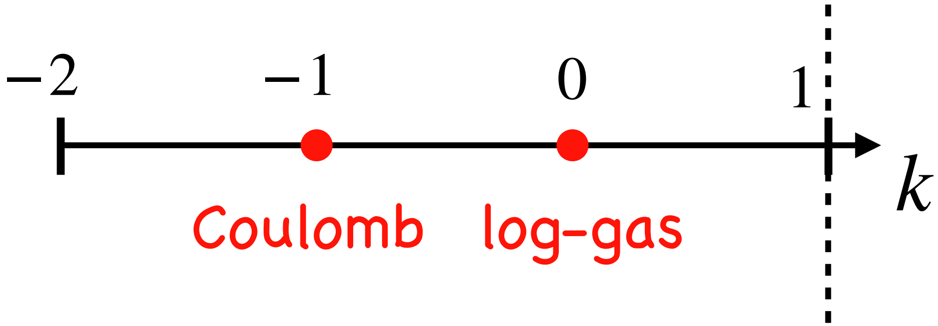

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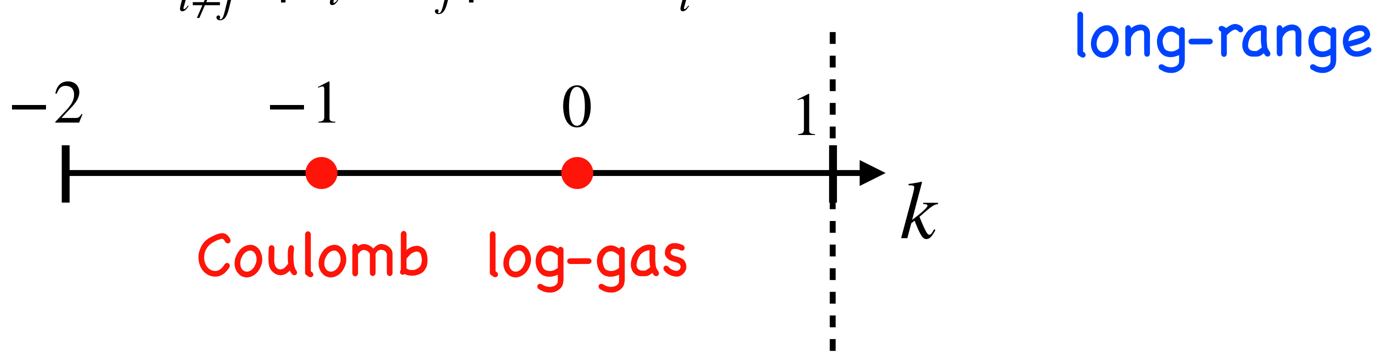

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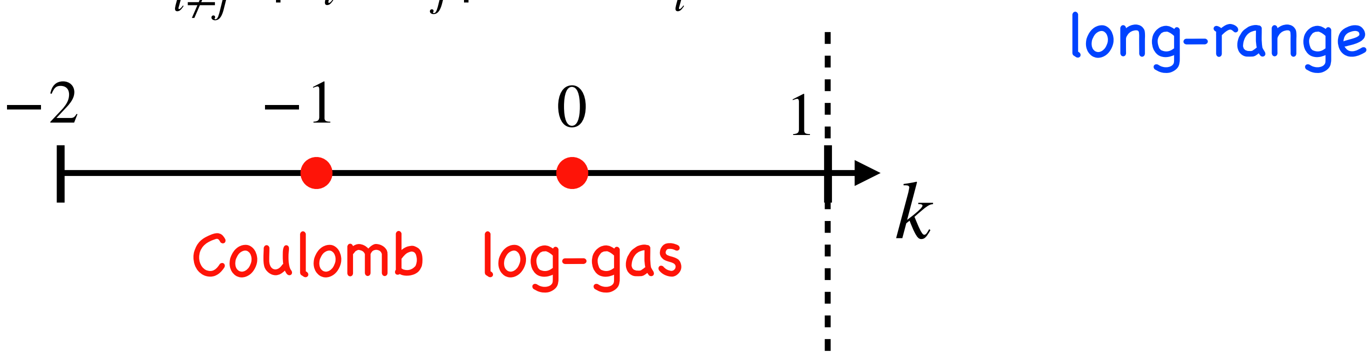

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$$\langle \rho_N(x) \rangle = \frac{1}{N} \sum_{i=1}^{N} \langle \delta(x - x_i) \rangle \xrightarrow[N \to \infty]{} \rho_{\mathrm{eq}}(x) = \frac{1}{\ell_0} F_k\left(\frac{x}{\ell_0}\right)$$

$$F_k(z) = A_k \left(1 - \frac{z^2}{4}\right)_+^{\frac{k+1}{2}}$$

S. Agarwal, A. Dhar, M. Kulkarni, A. Kundu, S. N. Majumdar, D. Mukamel, G. S. '19

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Linear statistics for the 1d Riesz gas

Le Doussal, G. S. '24

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$$p = k + 1 \qquad c_2 = \frac{\sin\left(\frac{\pi p}{2}\right) \Gamma\left(\frac{p}{2} + 1\right)^2}{\pi J p |k| \Gamma(p+2)} \quad , \quad \frac{c_m}{c_2} = \frac{2^{3-m} \Gamma\left(\frac{m+1}{2}\right) \Gamma\left(\frac{p+3}{2}\right)}{\sqrt{\pi}\, \Gamma\left(\frac{1}{2}(m+p+1)\right)}$$

$$c_{2,2} = \frac{\sin\left(\frac{\pi p}{2}\right) \Gamma\left(\frac{p}{2}\right)^2}{32 \pi J |k| (p+1)(p+3) \Gamma(p)} \quad , \quad \frac{c_{n,m}}{c_{2,2}} = \frac{2^{-m-n+8} \Gamma\left(\frac{m+1}{2}\right) \Gamma\left(\frac{n+1}{2}\right) \Gamma\left(\frac{p+1}{2}\right) \Gamma\left(\frac{p+5}{2}\right)}{\pi m (m+n+p-1) \Gamma\left(\frac{1}{2}(m+p+1)\right) \Gamma\left(\frac{1}{2}(n+p-1)\right)}$$

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$$a(\lambda) := \frac{\lambda^{m+k+1}}{m+k-(m-n)\lambda^{n+k}}$$

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- One-dimensional Riesz gases: main results
- Sketch of the derivation
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Q: Cumulants of $\mathscr{L}_N = \sum_{i=1}^{N} f(|\mathbf{x}_i|)$?

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- A general method valid in any $d \geq 1$: « Coulomb gas » method

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where $$E(\mathbf{x}_1, \ldots, \mathbf{x}_N) = \sum_{1 \le i < j \le N} U_d(|\mathbf{x}_i - \mathbf{x}_j|) + N \sum_{k=1}^{N} V(|\mathbf{x}_k|)$$

- Cumulant generating function

$$\chi(s, N) = \log \langle e^{-N\, s\, \mathscr{L}_N} \rangle = \log \langle e^{-N\, s \sum_{i=1}^{N} f(|x_i|)} \rangle$$

$$= \log \frac{Z_{N,V_s}}{Z_{N,V}} \quad \text{where} \quad V_s = V + \frac{s}{\beta} f$$

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$$\partial_s \chi(s, N) = -N \left\langle \sum_{i=1}^{N} f(|\mathbf{x_i}|) \right\rangle_{V_s} \approx -N^2 \int_{0 \le |\mathbf{x}| \le R_s} d\mathbf{x}\, \rho_{\mathrm{eq},s}(|\mathbf{x}|) f(|\mathbf{x}|)$$

where $\rho_{\mathrm{eq},s}(|\mathbf{x}|) = \dfrac{1}{\Omega_d} \Delta \left(V(|\mathbf{x}|) + \dfrac{s}{\beta} f(|\mathbf{x}|) \right)$ for $0 \le |\mathbf{x}| \le R_s$

with $$R_s^{d-1} \left(V'(R_s) + \frac{s}{\beta} f'(R_s) \right) = 1$$

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where $\rho_{\mathrm{eq},s}(|\mathbf{x}|) = \dfrac{1}{\Omega_d} \Delta \left(V(|\mathbf{x}|) + \dfrac{s}{\beta} f(|\mathbf{x}|) \right)$

- More explicitly, using rotational invariance

$$\rho_{\mathrm{eq},s}(x = |\mathbf{x}|) = \frac{1}{\Omega_d} \frac{1}{x^{d-1}} \left(x^{d-1} V'(x) + \frac{s}{\beta} x^{d-1} f'(x) \right)'$$

$$\partial_s \chi(s,N) \simeq -N^2 \int_0^{R_s} dx\, f(x) \left(x^{d-1} V'(x) + \frac{s}{\beta} x^{d-1} f'(x) \right)'$$

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- Taking one more derivative

$$\boxed{\frac{1}{N^2} \partial_s^2 \chi(s,N) \underset{N \to \infty}{\longrightarrow} \beta^{-1} \int_0^{R_s} dx\, x^{d-1} \, [f'(x)]^2}$$

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- Coulomb gases in d-dimensions: main results
- One-dimensional Riesz gases: main results
- Sketch of the derivation
  - Linear statistics of d-dimensional Coulomb gas
  - Linear statistics of the (long-range) 1d Riesz gas
- Conclusion and perspectives

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- Some of these results for the Riesz gas can be extended to the counting statistics (short-range): nontrivial edge effects

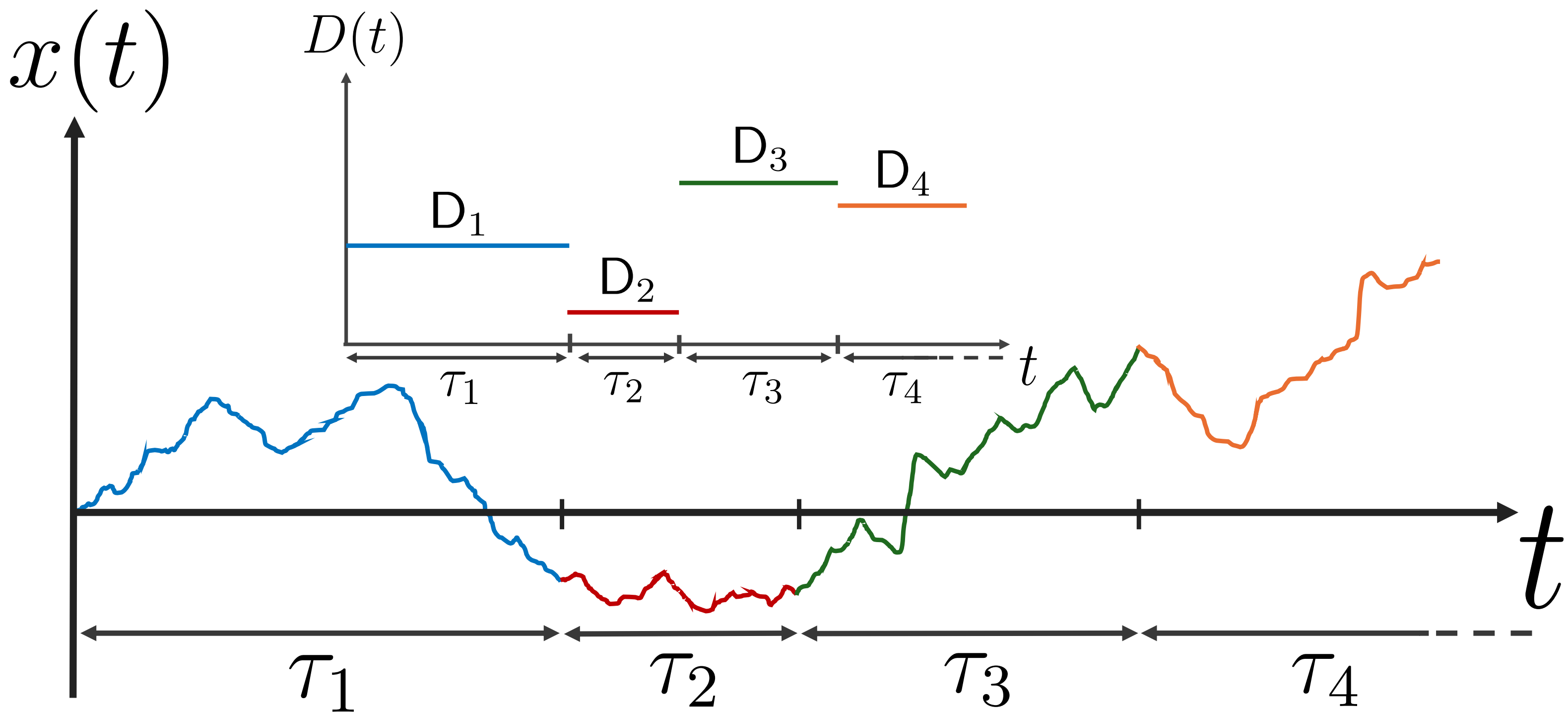
Thank You !

&

Happy Birthday

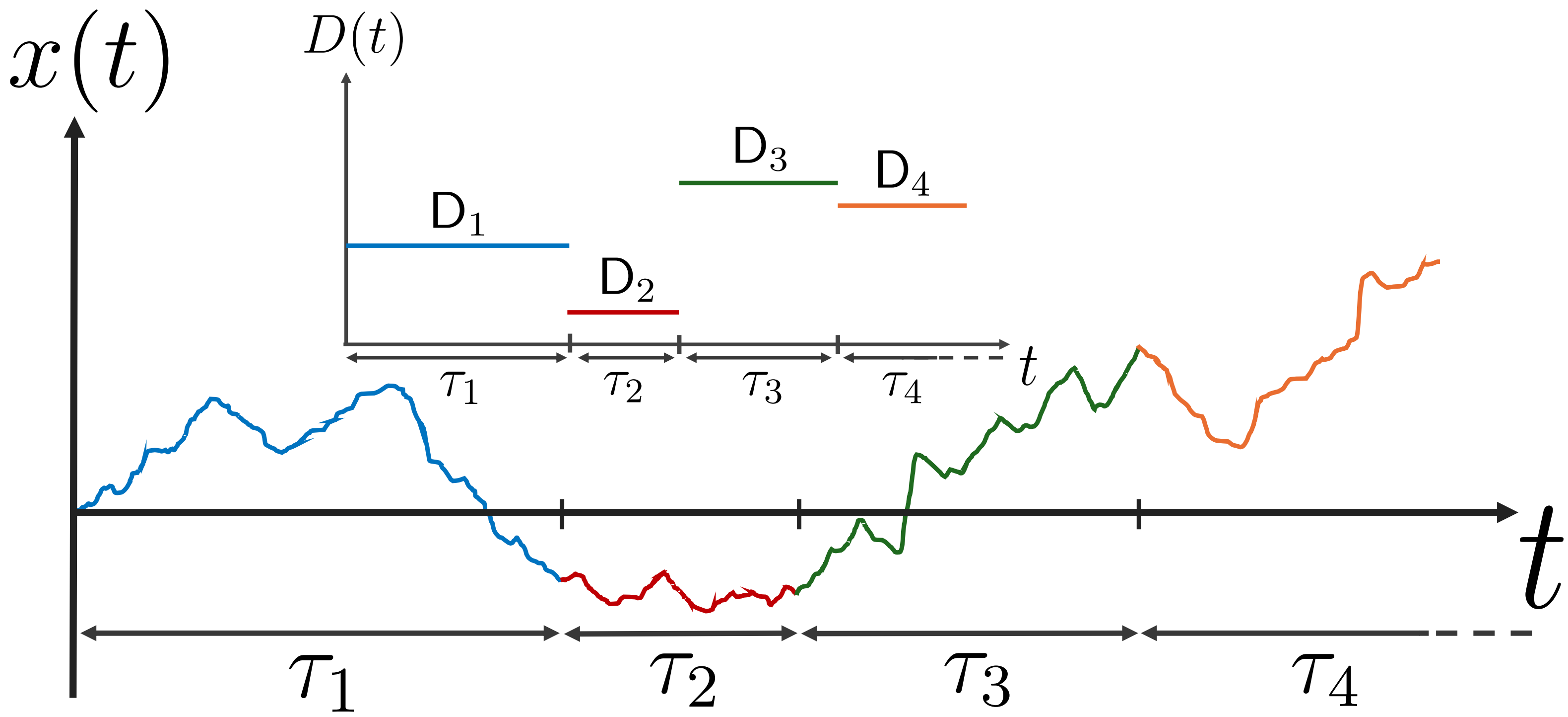
Peter !

Collaborative project: A simple diffusion model for free cumulants



Collaborative project: A simple diffusion model for free cumulants

- Thursday 11:00 am, Room 140 @ MATRIX House



M. Guéneau, S. N. Majumdar, G. S., Phys. Rev. Lett. **135**, 067102 (2025)

Two different methods to compute the cumulants

- A general method valid in any $d \geq 1$: « Coulomb gas » method

- The case $d = 2$ and $\beta = 2$ (GinUE) exploiting the determinantal structure

The case of normal matrices ($d = 2$, $\beta = 2$)

$$P(z_1, \ldots, z_N) = \frac{1}{Z_{N,V}} \prod_{i<j} |z_i - z_j|^2 \prod_{i=1}^{N} e^{-2NV(|z_i|)}$$

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- Generating function

$$\langle e^{-Ns\mathcal{L}_N} \rangle = \frac{1}{Z_{N,V}} \int d^2 z_1 \ldots d^2 z_N \prod_{j<k} |z_j - z_k|^2 \prod_{j=1}^{N} e^{-2N\,V(|z_j|) - N\,s\,f(|z_j|)}$$

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- Using Cauchy-Binet

$$\langle e^{-Ns\mathscr{L}_N} \rangle = \frac{N!}{Z_{N,V}} \det_{1 \le j,k \le N} \int d^2 z\, z^{j-1}\, \bar{z}^{k-1}\, e^{-2\,N\,V(|z|) - s\,N f(|z|)}$$

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$$\chi(s, N) = \log \left\langle e^{-s N \mathscr{L}_N} \right\rangle$$

$$\chi(s, N) = \sum_{\ell=0}^{N-1} \ln \left(2N^{1+\ell} \int_0^\infty dr \frac{r^{2\ell+1}}{\ell\,!} e^{-N r^2 - s N f(r)} \right)$$

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→ Asymptotic analysis for large N

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$\Longrightarrow$ Asymptotic analysis for large N

Similar asymptotic analysis has recently been performed for the counting statistics and for linear statistics in Ameur-Charlier-Cronvall-Lenells '22, Akemann-Byun-Ebke-GS '23, (see also Ameur-Charlier-Cronvall '22)

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→ For large N, the sum over ℓ is dominated by $\ell = O(N)$

The case of complex Ginibre matrices ($d = 2,\ \beta = 2$)

$$\chi(s,N) = \sum_{\ell=0}^{N-1} \ln\left(2N^{1+\ell}\int_0^\infty dr\,\frac{r^{2\ell+1}}{\ell\,!}e^{-N\,r^2 - s\,N f(r)}\right)$$

- For large N, saddle point analysis

$$\chi(s,N) \simeq N\int_0^1 d\lambda \ln\left(\frac{\sqrt{2N}}{\sqrt{\pi\lambda}}\int_0^\infty dr\, r\, e^{-N\,\phi_{s,\lambda}(r)}\right)$$

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- This yields

$$\chi(s,N) \simeq -N^2\int_0^1 d\lambda\, \min_{r\geq 0}[\phi_{s,\lambda}(r)] = -N^2 \int_0^1 d\lambda\, \phi_{s,\lambda}(r_{s,\lambda})$$

where $$\partial_r \phi_{s,\lambda}\Big|_{r=r_{s,\lambda}} = 0 \Longleftrightarrow r_{s,\lambda}^2 + \frac{s}{2} r_{s,\lambda} f'(r_{s,\lambda}) = \lambda$$

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- This can be generalized straightforwardly to complex normal matrices

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with arbitrary $V(|z|)$

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- This can also be generalized to symplectic Ginibre matrices (i.e., this property holds beyond the pure Coulomb gas)

$$\chi(s,N) = \log\left(\prod_{j=1}^{N} \frac{\int_0^{+\infty} dr\, r^{4j-1}\, e^{-2N\,r^2 - N\,sf(r)}}{\int_0^{+\infty} dr\, r^{4j-1}\, e^{-2Nr^2}}\right)$$

Rider '04,
Byun & Forrester '22