Generalized Meixner-type free gamma distributions

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Log-gases in Caeli Australi: Recent Developments in and around Random Matrix Theory

From this April our official university name was changed.

Osaka University $\rightarrow$ the University of Osaka

Abstract

In this talk, I will first explain my motivaton from probability theory and free probability. Subsequently, we introduce the class of generalized Meixner-type free gamma distributions. This class can be represented as a mixture of Marchenko-Pastur distribution in the sense of free multiplicative convolution.

This work is based on joint work with Yuki Ueda (Hokkaido University of Education, Asahikawa).

Motivation

Free probability

Main object and results

Motivation

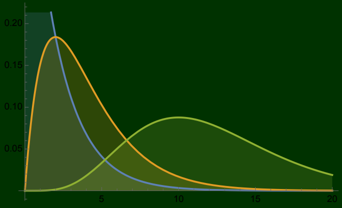
Motivation: the gamma distribution 1

The gamma distribution has the density function as follow:

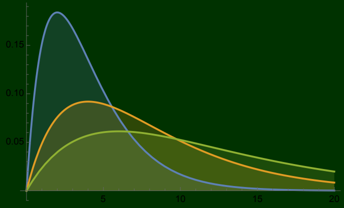
$$\gamma(t,\theta)(\mathrm{d}x) = \frac{1}{\theta^t \Gamma(t)} x^{t-1} e^{-\frac{x}{\theta}} \mathbf{1}_{(0,\infty)} \mathrm{d}x, \quad t, \theta > 0.$$

t: shape parameter, θ: mean parameter.

- The class of Gamma distributions include the chi-squera, exponential, Erlang distribuitons
- Huge range of application: queuing theory, non-life insurance, finance, econometrics, Bayesian statistics, life testing,....



$t = 1, 2, 6$ and $\theta = 1/2$



$t = 2$ and $\theta = 2, 4, 6$

Motivation: the gamma distribution 2 (infinite divisiblity(ID))

- μ is ID $\overset{\text{def}}{\iff} \forall n \in \mathbb{N}$, $\exists \mu_n \in \mathcal{P}$ s.t. $\mu = \mu_n^{*n}$.
- μ is ID iff $\exists 1 a \geqslant 0, \eta \in \mathbb{R}$, and Lévy meas. ν s.t.

$$\widehat{\mu}(z) = \exp\left[-\frac{az^2}{2} + i\eta z + \int_{(-\infty,\infty)} (e^{izx} - 1 - izx\mathbf{1}_{[-1,1]})(x)\nu(\mathrm{d}x)\right].$$

- We can construct Lévy processes from ID dist.
- All gamma distributions are ID:

$$\widehat{\gamma(t,\theta)}(z) = \left(\frac{1}{1 - i\theta z}\right)^t = \exp\left[\int_{(0,\infty)} (e^{izx} - 1)\frac{te^{-\frac{x}{\theta}}}{x}\mathrm{d}x\right].$$

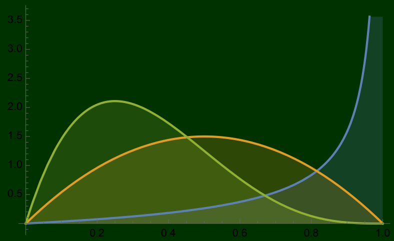
  Moreover it is selfdecomposable(SD).
  μ is SD $\overset{\text{def}}{\iff} \forall c \in (0,1)$, $\exists \rho_c$ s.t. $\mu = D_c(\mu) * \rho_c$, where
  $D_c(\mu)(B) = \mu(\frac{1}{c}B)$
- $\gamma(t,\theta) = \gamma(1,\theta)^{*t}$, $D_\theta(\gamma(t,1)) = \gamma(t,\theta)$

Motivation: the beta-gamma relation 1

Assume that X_1, X_2 are indep. and $X_1 \sim \gamma(\beta_1, 1)$, $X_2 \sim \gamma(\beta_2, 1)$.

- $\frac{X_1}{X_1+X_2}$ and $X_1 + X_2$ are independent and $\frac{X_1}{X_1+X_2}$ has the beta distribution:

$$f_{\frac{X_1}{X_1+X_2}}(x) = \frac{1}{B(\beta_1, \beta_2)} x^{\beta_1 - 1}(1-x)^{\beta_2 - 1}\mathbf{1}_{[0,1]}(x)$$

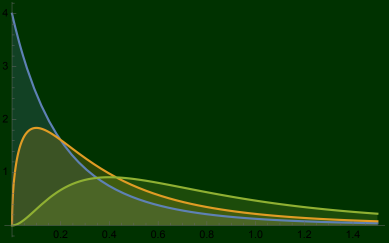


Motivation: the beta-gamma relation 1

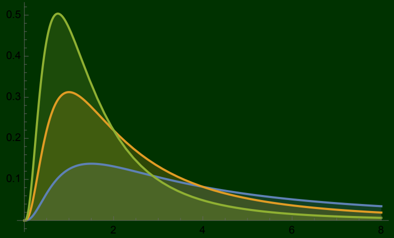
Assume that X_1, X_2 are indep. and $X_1 \sim \gamma(\beta_1, 1)$, $X_2 \sim \gamma(\beta_2, 1)$.

- $\frac{X_1}{X_2}$ has the beta distribution of second kind or beta prime distribution:

$$f_{\frac{X_1}{X_2}}(x) = \frac{1}{B(\beta_1, \beta_2)} x^{\beta_1 - 1} / (1+x)^{\beta_1+\beta_2} \mathbf{1}_{[0,\infty)}(x)$$



$\beta_1 = 1, 1.5, 3$ and $\beta_2 = 4$



$\beta_1 = 4$ and $\beta_2 = 1, 2, 3$

Free probability

Rough skech of NCP worlds

- Free prob. = non-commutative prob. + free indep.
- There are some beautiful correspondences between prob. and free prob. (or non-commutative prob.)
- the Gauss distribution $\leftrightarrow$ the semi-circule distribution
  - (Voiculescu) The theory on the sum of free indep. r.v.
    limit theorems: C.L.T. $\leftrightarrow$ free C.L.T.
  - (Speicher, Biane, Barndroff-Nielsen and Thorbjørnsen)
    The theory on stochastic processes:
    Brownian motion $\leftrightarrow$ free Brownian motion,
    Lévy processes $\leftrightarrow$ free Lévy processes.
  - (Biane and Speicher, Peccati, Kemp, Nourdin and Speicher)
    Stochastic analysis:
    Wiener Chaos $\leftrightarrow$ Wigner Chaos,
    Malliavin calculus and Stein methods.
  - Contribution to physics??: Schwinger-Dyson equation, ...

Free independence, Free additive convolution

- Non-commutative probability: a probability model with non-commutativity for the product of random variables. $XY \neq YX$
- Free probability = non-commutative probability + free independence
- $K = \mathbb{R}$ or $\mathbb{R}_+ := [0, \infty)$.
- $\mathcal{P}(K) := \{\mu \mid \text{Borel prob. meas. on } K\}$
- $\boxplus$: Free additive convolution
  X, Y: free independent r.v. distributed as. $X \sim \mu$, $Y \sim \rho$.
  $\mu \boxplus \rho$ is the distribution of $X + Y$.
- μ is free infinitely divisible if $\forall n \in \mathbb{N}$, $\exists \mu_n \in \mathcal{P}$ s.t. $\mu = \mu_n^{\boxplus n}$.
  $I(\boxplus) := \{\mu \mid \text{free infinitely divisible on } \mathbb{R}\}$
-
    - Analytic method (the Cauchy transform, R-transform, subordination)
    - Combinatorial method (free cumulant sequences)

Analytic tools

- The Cauchy transform $G_\mu : \mathbb{C}^+ \to \mathbb{C}^-$ of $\mu \in \mathcal{P}(\mathbb{R})$;

$$G_\mu(z) := \int_{\mathbb{R}} \frac{1}{z-t}\mu(\mathrm{d}t), \quad z \in \mathbb{C}^+$$
$$\left(= \frac{1}{z}\sum_{k=0}^{\infty}\int_{\mathbb{R}} \frac{t^k}{z^k}\mu(\mathrm{d}t), \quad z \in \mathbb{C}^+ \right).$$

- The reciprocal Cauchy transform $F_\mu : \mathbb{C}^+ \to \mathbb{C}^+$ of $\mu \in \mathcal{P}(\mathbb{R})$;

$$F_\mu(z) := 1/G_\mu(z), \quad z \in \mathbb{C}^+.$$

  It is the Pick function.

Pick function

- $\mathcal{M}_{\mathrm{f}} :=$ {finite Borel meas. on $\mathbb{R}$}

Theorem 2.1

$F : \mathbb{C}^+ \to \mathbb{C}^+$ *is analytic with*

$$\lim_{y\to\infty} \frac{F(iy)}{iy} = 1.$$

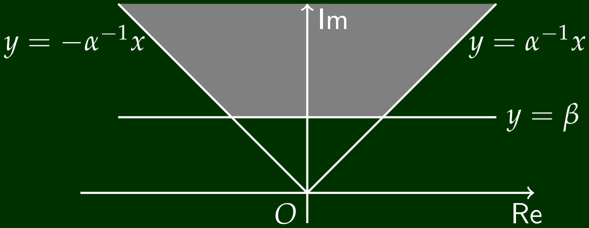
$\iff$ $\exists 1 \mu \in \mathcal{P}(\mathbb{R})$ *s.t.* $F = F_\mu$ *on* $\mathbb{C}^+$
$\iff$ $\exists 1 a \in \mathbb{R}, \rho \in \mathcal{M}_{\mathrm{f}}$ *s.t.* $F = F_\mu$ *on* $\mathbb{C}^+$.

$$F(z) = a + z + \int_{-\infty}^{\infty} \frac{1+tz}{t-z} \rho(\mathrm{d}t), \quad z \in \mathbb{C}^+$$

Analytic tools 2

- F_μ is univalent on $\Gamma_{\alpha,\beta}$. So that inverse function $F_\mu^{\langle -1 \rangle}$ of F_μ can be defined.

$$\Gamma_{\alpha,\beta} := \{z = x + iy : y > \beta, |x| < \alpha y\}$$



$\alpha > 0, \beta > 0$ depend on the distribution μ.

Analytic tools 2

- F_μ is univalent on $\Gamma_{\alpha,\beta}$. So that inverse function $F_\mu^{\langle -1\rangle}$ of F_μ can be defined.

$$\Gamma_{\alpha,\beta} := \{z = x + iy : y > \beta, |x| < \alpha y\}$$

- The R-transform $R_\mu : \mathbb{C}^- \to \mathbb{C}$ of $\mu \in \mathcal{P}$;

$$R_\mu(z) = zF_\mu^{\langle -1\rangle}(z^{-1}) - 1, \quad z^{-1} \in \Gamma_{\alpha,\beta}$$

- $\mu \in \mathcal{P}(\mathbb{R})$ is FID $\iff \exists 1 a \geqslant 0, \gamma \in \mathbb{R}, \nu$: Lévy measure on $\mathbb{R}\backslash\{0\}$ s.t.

$$R_\mu(z) := az^2 + \gamma z + \int_{\mathbb{R}\backslash\{0\}} \left(\frac{1}{1 - zt} - 1 - zt\mathbb{1}_{[-1,1]}(t) \right) \nu(\mathrm{d}t), \tag{1}$$

$z \in \mathbb{C}^-$.

Example: semi-circle distribution

In (1), we set $a = \sigma^2$, $\gamma = 0$ and $\nu = 0$: From definition and

$$R_\mu(\omega) = \sigma^2\omega^2,$$

we have

$$F^{\langle -1\rangle}_{\mathbf{S}(0,\sigma^2)}(\omega) = \frac{\sigma^2}{\omega} + \omega.$$

Then we set $\omega = F_{\mathbf{S}(0,\sigma^2)}(z)$ and obtain

$$z = \frac{\sigma^2}{F_{\mathbf{S}(0,\sigma^2)}(z)} + F_{\mathbf{S}(0,\sigma^2)}(z) \iff F_{\mathbf{S}(0,\sigma^2)}(z)^2 - zF_\mu(z) + \sigma^2 = 0.$$

Example: semi-circle distribution

In (1), we set $a = \sigma^2$, $\gamma = 0$ and $\nu = 0$:

$$z = \frac{\sigma^2}{F_{\mathbf{S}(0,\sigma^2)}(z)} + F_{\mathbf{S}(0,\sigma^2)}(z) \iff F_{\mathbf{S}(0,\sigma^2)}(z)^2 - zF_\mu(z) + \sigma^2 = 0.$$

Finally, we get its Cauchy transform:

$$F_{\mathbf{S}(0,\sigma^2)}(z) = \frac{z + \sqrt{z^2 - 4\sigma^2}}{2}, \quad G_{\mathbf{S}(0,\sigma^2)}(z) = \frac{z - \sqrt{z^2 - 4\sigma^2}}{2}.$$

By Stieltjes inversion formula, the density function of $\mathbf{S}(0, \sigma^2)$ is

$$f(x) = -\frac{1}{\pi} \lim_{y \downarrow 0} \mathrm{Im} G_\mu(x + iy) = \frac{1}{\pi} \sqrt{4\sigma^2 - x^2} \mathbb{1}_{[-2\sigma, 2\sigma]}(x).$$

Example: the standard semi-circle distribution

Conversely we can compute the Cauchy transform of semi-circle distribution by its moment sequence. First, we can compute easily

$$m_k = \int_{\mathbb{R}} x^k \frac{1}{\pi}\sqrt{4-x^2}\mathbb{1}_{[-2,2]}(x)$$
$$= \begin{cases} \frac{1}{p+1}\binom{2p}{p} =: \mathrm{Cat}(p) & k = 2p \\ 0 & k = 2p+1 \end{cases}.$$

Here we know the recursive formula

$$\mathrm{Cat}(p+1) = \sum_{i=0}^{p} \mathrm{Cat}(i)\mathrm{Cat}(p-i),$$

and we can compute its generating function and Cauchy transform via

$$\sum_{n=0}^{\infty} z^{2n} m_{2n} = \sum_{n=0}^{\infty} z^{2n}\mathrm{Cat}(n) = \frac{1-\sqrt{1-4z^2}}{2z^2}.$$

Example: Marchenko-Pastur distribution(free Poisson distribution)

Let's consider the following:

$$R_{\pi_{\lambda,1}}(z) = \lambda z + \int_{\mathbb{R}\setminus\{0\}} \left(\frac{1}{1-zt} - 1 - zt\mathbb{1}_{[-1,1]}(t) \right) \lambda\delta_1(\mathrm{d}t),$$
$$= \frac{\lambda z}{1-z} = zF^{\langle -1 \rangle}_{\pi_{\lambda,1}}(z^{-1}) - 1$$

So we have

$$F^{\langle -1 \rangle}_{\pi_{\lambda,1}}(\omega) = \frac{\lambda\omega}{\omega - 1} + \omega$$

Its the F-transform is $F_{\pi_{\lambda,1}}(z) = \frac{(z+1-\lambda)-\sqrt{(z+1-\lambda)^2-4z}}{2}$. Thus, we obtain the Cauchy transform $G_{\pi_{\lambda,1}}(z) = \frac{(z+1-\lambda)+\sqrt{(z+1-\lambda)^2-4z}}{2z}$. By Stieltjes inversion formula, the density function of $\pi_{\lambda,1}$ is

$$\pi_{\lambda,1}(\mathrm{d}x) = \max\{1-\lambda, 0\}\delta_0(\mathrm{d}x) + \frac{1}{2\pi}\frac{\sqrt{4\lambda - (x-1-\lambda)^2}}{x}\mathbf{1}_{[(1-\sqrt{\lambda})^2,(1+\sqrt{\lambda})^2]}(x)\mathrm{d}x.$$

Compare with classical probability theory

Proposition 2.2

$\mu \in I(\boxplus)$ ($\mu \in I(*)$) has Lévy-Khintchine representation: there exist unique $a \geqslant 0$, $\gamma \in \mathbb{R}$, and Lévy meas. ν such that

$$R_\mu(z) = az^2 + \gamma z + \int_{\mathbb{R}} \left(\frac{1}{1-xz} - 1 - xz1_{[-1,1]}(x) \right) \nu(dx).$$

$$\left(\log \left(\int e^{izx} \mu(\mathrm{d}x) \right) = -\frac{1}{2} az^2 + i\gamma z + \int_{\mathbb{R}} \left(e^{izx} - 1 - izx1_{[-1,1]}(x) \right) \nu(dx). \right)$$

Remark 2.3

a is the semi-circle part, γ is shift part. (a, ν, γ) is called free characteristic triplet.

Bercovici-Pata bijection

$\Lambda : I(*) \overset{\text{bij}}{\to} I(\boxplus)$ is a map from $\mu \in I(*)$ with the characteristic triplet (a, ν, γ) to $\Lambda(\mu) \in I(\boxplus)$ with the free characteristic triplet (a, ν, γ).

It is called **Bercovici-Pata bijection**.

the normal distr. $\mapsto$ the semi-circle distr.

the Poisson distr. $\mapsto$ the Marchenko-Pastur distr.

the Cauchy distr. $\mapsto$ the Cauchy distr.

Main object and results

The Meixner distributions 1

Let $\nu_{s,a,b}$ be a probability measure whose Cauchy transform is given by

$$\int_{\mathbb{R}} \frac{1}{z-x}\nu_{s,a,b}(\mathrm{d}x) = \cfrac{1}{z - \cfrac{s}{z-a-\cfrac{s+b}{z-a-\cfrac{s+b}{\cdots}}}} \tag{2}$$

$$= \frac{(s+2b)z + sa - s\sqrt{(z-a)^2 - 4(s+b)}}{2(bz^2+saz+s^2)}, \tag{3}$$

where $a \in \mathbb{R}$, $b \geq -1$ and $s \geq 0$. The three-parameter measure $\nu_{s,a,b}$ is called the *centered free Meixner distribution*.

The Meixner distributions 2

Integral representation for the R-transform of $\nu_{s,a,b}$ is given by

$$R_{\nu_{s,a,b}}(z) = \int_{\mathbb{R}} \frac{z^2}{1 - xz} s\ w_{a,b}(x) \mathrm{d}x,$$

where

$$w_{a,b}(x) = \frac{1}{2\pi b} \sqrt{4b - (x-a)^2}\,\mathbf{1}_{[a-2\sqrt{b}, a+2\sqrt{b}]}(x)$$

is the density of Wigner's semicircle law with mean $a \in \mathbb{R}$ and variance $b \geq 0$.

The Meixner-type free gamma distributions

The *Meixner-type free gamma distribution* is given by

$$R_{\eta(t,\theta)}(z) = \int_{(0,\infty)} \left(\frac{1}{1-zx} - 1 \right) \frac{t\theta k_{\theta,1}(x)}{x} \mathrm{d}x = \frac{t}{2}(1 - \sqrt{1-4\theta z}), \quad z \in \mathbb{C}^-,$$

where $t > 0$ is the time (shape) parameter, $\theta > 0$ is the scale parameter and the function $k_{\theta,\lambda}$ is

$$k_{\theta,\lambda}(x) := \frac{\sqrt{(a^+ - x)(x - a^-)}}{2\pi\theta x} \mathbf{1}_{(a^-,a^+)}(x), \qquad \lambda \geq 1,$$

where $a^{\pm} := \theta(\sqrt{\lambda} \pm 1)^2$, respectively. We can easily see that

- $\eta(t,\theta)$ is freely infinitely divisible for all $t, \theta > 0$;
- $\eta(t,\theta) = \eta(1,\theta)^{\boxplus t}$ for all $t, \theta > 0$;
- $\eta(t_1,\theta) \boxplus \eta(t_2,\theta) = \eta(t_1 + t_2, \theta)$ for all $t_1, t_2, \theta > 0$.

Generalized Meixner-type free gamma distributions

The family $\{\mu_{t,\theta,\lambda} : t, \theta > 0, \lambda \geq 1\}$ of probability measures whose R-transforms are defined by

$$R_{\mu_{t,\theta,\lambda}}(z) = \int_{(0,\infty)} \left(\frac{1}{1-zx} - 1\right) \frac{tk_{\theta,\lambda}(x)}{x} \mathrm{d}x, \qquad z \in \mathbb{C}^-$$
$$= t \cdot \frac{1 + \theta(1-\lambda)z - \sqrt{(1+\theta(1-\lambda)z)^2 - 4\theta z}}{2\theta}.$$

We call the measures $\mu_{t,\theta,\lambda}$ *generalized Meixner-type free gamma distributions*.

Remark 3.1

If we replace t, λ by $t\theta$, 1 respectively, we get the Meixner-type free gamma distributions $\eta(t,\theta)$.

It is completely different from free gamma distribution by Pérez-Abreu and S.

Atom and density

Proposition 3.2

Consider $t, \theta > 0$ and $\lambda \geq 1$. Then

$$\mu_{t,\theta,\lambda}(\mathrm{d}x) = \max\left\{0, \frac{\theta(\lambda-1)-t}{\theta(\lambda-1)}\right\}\delta_0 + \frac{t\sqrt{(x-\alpha^-)(\alpha^+-x)}}{2\pi\theta x(x+t(\lambda-1))}\mathbf{1}_{[\alpha^-,\alpha^+]}(x)\mathrm{d}x,$$

where $\alpha^\pm := \theta(\lambda+1)+t\pm 2\sqrt{\theta\lambda(\theta+t)}$.

Distributional properties

Theorem 3.3

Let $t, \theta > 0$ and $\lambda \geq 1$. Then the following properties hold.

(1) *The measure $\mu_{t,\theta,\lambda}$ is freely selfdecomposable if and only if $\lambda = 1$. In addition, it is unimodal if and only if $1 \leq \lambda \leq 1 + t/\theta$.*

(2)

$$\mu_{t,\theta,\lambda} = D_{t(\lambda-1)}\left(\pi_{1,\frac{t}{\theta(\lambda-1)}} \boxtimes (\pi_{1,1+\frac{t}{\theta}})^{\langle -1\rangle}\right) = \mu_{t,\theta,1} \boxtimes \pi_{q^{-1},q}, \quad \lambda > 1.$$

Here, $q = \frac{t}{\theta(\lambda-1)}$ and $\pi_{\theta,\lambda}$ is the probability measure defined by

$$\pi_{\theta,\lambda}(\mathrm{d}x) = \max\{0, 1-\lambda\}\delta_0 + k_{\theta,\lambda}(x)\,\mathrm{d}x.$$

(3) $\mu_{t,\theta,1+t/\theta} = \mu_{t,\theta,1} \boxtimes \pi_{1,1}$. *$\mu_{t,\theta,1+t/\theta}$ is a free compound Poisson distributions.*

The S-transform 1

The S–transforms of ρ are defined by

$$S_\rho(z) = \frac{z+1}{z}\Psi_\rho^{-1}(z), \quad z \in \Psi_\rho(i\mathbb{C}^+),$$

where $\Psi_\rho^{-1}(z)$ is the inverse of $\Psi_\rho(z) = \int_{(0,\infty)} \frac{xz}{1-xz}\mu(\mathrm{d}x)$ with respect to composition.

Proposition 3.4

For $\rho_1 \in \mathcal{P}_+$ and $\rho_2 \in \mathcal{P}_+$, which are not δ_0, there exist $\alpha > 0$ and $\beta > 0$ such that

$$S_{\rho_1 \boxtimes \rho_2}(z) = S_{\rho_1}(z)S_{\rho_2}(z), \quad z \in \Psi_{\rho_1}(i\mathbb{C}^+) \cap \Psi_{\rho_2}(i\mathbb{C}^+),$$
$$S_{\rho_1^{\boxplus t}}(z) = \frac{1}{t}S_{\rho_1}\left(\frac{z}{t}\right).$$
$$R_\mu(zS_\mu(z)) = z$$

For $c > 0$, the dilation operator D_c on $\mathcal{P}$ is defined by $D_c(\mu)(B) := \mu\left(\frac{1}{c}B\right)$ for any Borel set B on $\mathbb{R}_+$, where $\frac{1}{c}B = \{x \in \mathbb{R}; \frac{1}{c}x \in B\}$.

We have

$$S_{D_c(\mu)}(z) = \frac{1}{c}S_\mu(z).$$

The S-transform 2

$$S_{\pi_{\theta,\lambda}}(z) = \frac{1}{\theta(\lambda+z)}, \qquad z \text{ in a neighborhood of } (-1+\pi_{\theta,\lambda}(\{0\}),0).$$

$\mu^{\langle c\rangle}$ is the pushforward of μ by the map $(0,\infty) \ni x \mapsto x^c$, for all $c \neq 0$ and for all probability measures μ on $(0,\infty)$.

$$S_{\mu^{\langle -1\rangle}}(z) = \frac{1}{S_\mu(-z-1)}, \qquad z \in (-1,0).$$

$$S_{\pi_{\theta,\lambda}^{\langle -1\rangle}}(z) = \frac{1}{S_{\pi_{\theta,\lambda}}(-z-1)} = \theta(\lambda-1-z), \qquad z \in (-1,0).$$

Proof

From direct computation,

$$S_{\mu_{t,\theta,\lambda}}(z) = \frac{R^{\langle -1\rangle}_{\mu_{t,\theta,\lambda}}(z)}{z} = \frac{t-\theta z}{t(t+\theta(\lambda-1)z)}$$
$$= \frac{1}{t(\lambda-1)}\frac{1}{\frac{t}{\theta(\lambda-1)}+z}\left(\frac{t}{\theta}-z\right)$$
$$= S_{D_{t(\lambda-1)}\left(\pi_{1,\frac{t}{\theta(\lambda-1)}}\boxtimes(\pi_{1,1+\frac{t}{\theta}})^{\langle -1\rangle}\right)}(z)$$

Recall

$$S_{\rho_1\boxtimes\rho_2}(z) = S_{\rho_1}(z)S_{\rho_2}(z); \qquad S_{D_c(\mu)}(z) = \frac{1}{c}S_\mu(z);$$
$$S_{\pi_{\theta,\lambda}}(z) = \frac{1}{\theta(\lambda+z)};$$
$$S_{\pi^{\langle -1\rangle}_{\theta,\lambda}}(z) = \frac{1}{S_{\pi_{\theta,\lambda}}(-z-1)} = \theta(\lambda-1-z).$$

Proof 2

Moreover,

$$\begin{aligned}
\mu_{t,\theta,\lambda} &= D_{t(\lambda-1)}\left(\pi_{1,\frac{t}{\theta(\lambda-1)}} \boxtimes (\pi_{1,1+\frac{t}{\theta}})^{\langle -1\rangle}\right) \\
&= D_{t(\lambda-1)} \circ D_{\frac{t}{\theta(\lambda-1)}}\left(\pi_{\frac{\theta(\lambda-1)}{t},\frac{t}{\theta(\lambda-1)}} \boxtimes (\pi_{1,1+\frac{t}{\theta}})^{\langle -1\rangle}\right) \\
&= \pi_{q^{-1},q} \boxtimes (\pi_{\frac{\theta}{t^2},1+\frac{t}{\theta}})^{\langle -1\rangle} \\
&= \pi_{q^{-1},q} \boxtimes \mu_{t,\theta,1},
\end{aligned}$$

Relation with Yoshida's results

Yoshida introduced the free beta prime distribution $f\beta'(a,b) = \pi_a \boxtimes \pi_b^{-1}$.

Theorem 3.5

Let us consider $a > 0$ *and* $b > 1$.

(1) *We have* $f\beta'(a,b) = \mu_{\frac{a}{b-1}\cdot\frac{a}{(b-1)^2},\frac{a+b-1}{a}}$.

(2) *The free Lévy measure of* $f\beta'(a,b)$ *is given by*

$$\frac{a}{b-1}\frac{k_{u,v}(x)}{x}\mathrm{d}x,$$

where $u = \frac{a}{(b-1)^2}$ *and* $v = \frac{a+b-1}{a}$.

(3) *The measure* $f\beta'(a,b)$ *is not freely selfdecomposable. In addition, it is unimodal if and only if* $a \geq 1$.

Potential

We can compute the Potential function:

$$(\mathcal{H}\mu)(x) := \lim_{\varepsilon\to+0} \mathrm{Re}\left(G_\mu(x+i\varepsilon)\right) = \frac{1}{2}V'(x), \qquad x \in \mathsf{supp}(\mu).$$

For $t,\theta > 0$ and $\lambda \geq 1$, we consider the potential functions $V_{t,\theta,\lambda}$ defined by

$$V_{t,\theta,\lambda}(x) := \begin{cases} \left(2+\frac{t}{\theta}\right)\log x + \frac{t^2}{\theta x}, & \lambda = 1, \\ \left(1-\frac{t}{\theta(\lambda-1)}\right)\log x \\ \qquad + \left(1+\frac{t\lambda}{\theta(\lambda-1)}\right)\log(x+t(\lambda-1)), & \lambda > 1. \end{cases}$$

From this potential function, we can consider the Gibbs measure:

$$\rho_{t,\theta,\lambda}(\mathrm{d}x) = \frac{1}{\mathcal{Z}_{t,\theta,\lambda}} \exp\{-V_{t,\theta,\lambda}(x)\}\mathrm{d}x \quad \text{for} \quad t,\theta > 0,\ \lambda \geq 1,$$

Explicit form

For $\lambda = 1$,

$$\rho_{t,\theta,1}(\mathrm{d}x) = \frac{\left(\frac{t^2}{\theta}\right)^{1+\frac{t}{\theta}}}{\Gamma(1+\frac{t}{\theta})} x^{-(2+\frac{t}{\theta})} e^{-\frac{t^2}{\theta x}} \mathrm{d}x, \tag{4}$$

It is the inverse gamma distribution with parameter (α, β): $\frac{\beta^\alpha}{\Gamma(\alpha)}(1/x)^{\alpha+1}\exp(-\beta/x)$.

$$\rho_{t,\theta,\lambda}(\mathrm{d}x) = \frac{(t(\lambda-1))^{\frac{t}{\theta}+1}}{B(\frac{t}{\theta(\lambda-1)}, 1+\frac{t}{\theta})} x^{-1+\frac{t}{\theta(\lambda-1)}} (x+t(\lambda-1))^{-1-\frac{t\lambda}{\theta(\lambda-1)}} \mathrm{d}x, \tag{5}$$

$$\lambda > 1.$$

It is the beta prime distribution or beta distribution of 2nd kind.

Classical counterpart

Theorem 3.6

Let us consider $t, \theta > 0$ *and* $\lambda \geq 1$.

(1) *The measure* $\rho_{t,\theta,\lambda}$ *is selfdecomposable, and therefore it is unimodal.*

(2) *For* $\lambda > 1$, *we have*

$$\rho_{t,\theta,\lambda} = D_{t(\lambda-1)}\left(\gamma\left(\frac{t}{\theta(\lambda-1)}, 1\right) \circledast \gamma\left(1 + \frac{t}{\theta}, 1\right)^{\langle -1 \rangle}\right)$$
$$= \rho_{t,\theta,1} \circledast \gamma\left(\frac{t}{\theta(\lambda-1)}, \frac{\theta(\lambda-1)}{t}\right).$$

(3) *In particular,* $\rho_{t,\theta,1+t/\theta} = \rho_{t,\theta,1} \circledast \gamma(1,1)$. *Hence the measure* $\rho_{t,\theta,1+t/\theta}$ *belongs to the class of mixture of exponential distributions.*

Free entropy maximizer

Theorem 3.7

For $t, \theta > 0$ and $1 \leq \lambda < 1 + t/\theta$, we have

$$\mu_{t,\theta,\lambda} = \operatorname{argmax}\{\Sigma_{V_{t,\theta,\lambda}}(\mu) : \mu \text{ is a p. m. on } (0,\infty)\},$$

where $\Sigma_{V_{t,\theta,\lambda}}(\mu)$ is the (Voiculescu's) free entropy:

$$\Sigma_{V_{t,\theta,\lambda}}(\mu) = \iint \log|x-y|\mu(\mathrm{d}x)\mu(\mathrm{d}y) - \int V_{t,\theta,\lambda}(x)\mu(\mathrm{d}x).$$

Corollary 3.8

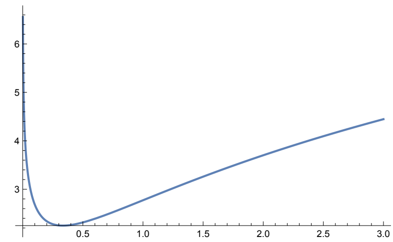
In particular, for $a, b > 1$, the measure $f\beta'(a,b)$ is the unique maximizer of the free entropy $\Sigma_{V_{a,b}}$, where

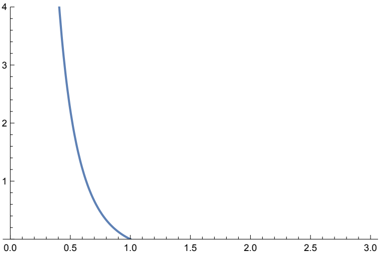
$$V_{a,b}(x) = (1-a)\log x + (a+b)\log(1+x), \qquad x > 0.$$

The graph of potential $V_{2,2}$ and its 2nd derivative

$$V_{2,2}(x) = -\log x + 4\log(1+x), \qquad x > 0.$$

$$V''_{2,2}(x) = \frac{1}{x^2} - \frac{4}{(x+1)^2}.$$





Thank you for your attention

and

Happy Birthday Peter!