Universality for Fluctuations of Counting Statistics of Random Normal Matrices

Joint work with Jordi Marzo and Joaquim Ortega-Cerdà
arXiv:2508.04386
(and Gernot Akemann and Maurice Duits, arXiv:2412.15854)

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- Given $Q : \mathbb{C} \to \mathbb{R}$ the associated random normal matrix model consists of all complex $n \times n$ normal matrices M, distributed by

$$d\mathscr{P}_n(M) = \frac{1}{\mathscr{Z}_n} e^{-n \operatorname{Tr} Q(M)} dM, \qquad dM = \prod_{1 \le j,k \le n} dA(M_{jk})$$

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- The eigenvalues describe the locations of n particles in a 2D Coulomb gas at inverse temperature 2, confined by the potential Q.
- The corresponding eigenvalues $z_1, \ldots, z_n \in \mathbb{C}$ of M are distributed as

$$d\mathbb{P}_n(z_1, \ldots, z_n) = \frac{1}{Z_n} \prod_{1 \le i < j \le n} |z_i - z_j|^2 \prod_{j=1}^{n} e^{-nQ(z_j)} dA(z_j).$$

- The eigenvalues form a determinantal point process: the k-point correlation functions can be expressed as $\rho_{n,k}(z_1, \ldots, z_k) =$

$$\frac{n!}{(n-k)!} \int_{\mathbb{C}^{n-k}} \frac{1}{Z_n} \prod_{1 \le i < j \le n} |z_i - z_j|^2 \prod_{j=1}^{n} e^{-nQ(z_j)}\, dA(z_{k+1}) \cdots dA(z_n)$$
$$= \det \left(\mathscr{K}_n(z_i, z_j) \right)_{1 \le i,j \le k}, \qquad k = 1, \ldots, n,$$

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- We make the Hermitian symmetric choice:

$$\mathscr{K}_n(z, w) = e^{-\frac{1}{2} n (Q(z) + Q(w))} \sum_{j=0}^{n-1} p_j(z) \overline{p_j(w)},$$

where the $p_j : \mathbb{C} \to \mathbb{C}$ are planar orthogonal polynomials (with degree j and positive leading coefficient)

$$\int_{\mathbb{C}} p_j(z) \overline{p_k(z)} e^{-nQ(z)} dA(z) = \delta_{j,k}, \qquad j, k = 0, 1, \ldots$$

- The eigenvalues accumulate on a compact set S_Q called the droplet.

$$\lim_{n\to\infty}\frac{1}{n}\rho_{n,1}(z,z)=\lim_{n\to\infty}\frac{1}{n}\mathscr{K}_n(z,z)=\begin{cases}\Delta Q(z), & z\in\mathring{S}_Q,\\ \frac{1}{2}\Delta Q(z), & z\in\partial S_Q,\\ 0, & z\in S_Q^c,\end{cases}$$

where $\Delta=\partial_z\overline{\partial}_z=\frac{1}{4}(\partial_x^2+\partial_y^2)$ denotes the quarter Laplacian.

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- In fact, we have $\frac{1}{n}\mathscr{K}_n(z,z)dA(z) \to d\mu_Q^*(z)$ in distribution where $d\mu_Q^*(z) = \chi_{S_Q}(z)\Delta Q(z)dA(z)$ minimizes the (energy) functional

$$I_Q(\mu) = \iint_{\mathbb{C}^2} \log\frac{1}{|z-w|} d\mu(z)d\mu(w) + \int_{\mathbb{C}} Q(z)d\mu(z)$$

over all compactly supported Borel probability measures on $\mathbb{C}$.

- (bulk) When $z_0 \in \mathring{S}_Q$, the Ginibre kernel arises as a scaling limit

$$\lim_{n\to\infty} \frac{1}{n\Delta Q(z_0)} \mathscr{K}_n \left(z_0 + \frac{\xi}{\sqrt{n\Delta Q(z_0)}}, z_0 + \frac{\eta}{\sqrt{n\Delta Q(z_0)}} \right) \equiv e^{-\frac{1}{2}|\xi|^2 - \frac{1}{2}|\eta|^2 - \xi\overline{\eta}}.$$

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- (edge) When $z_0 \in \partial S_Q$, the erfc kernel arises as a scaling limit

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- **Question 1**: given a set A, how many eigenvalues of M are in A?
- **Question 2**: how does this number fluctuate?

- In general the expectation and variance of $\Sigma_n[f]$ is given by

$$\mathbb{E}\Sigma_n[f] = \int_{\mathbb{C}} f(z)\mathscr{K}_n(z,z)dA(z)$$
$$\operatorname{Var}\Sigma_n[f] = \frac{1}{2}\int_{\mathbb{C}^2} (f(z)-f(w))^2|\mathscr{K}_n(z,w)|^2 dA(z)dA(w).$$

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- For counting statistics the number variance is given by

$$\operatorname{Var} N_A^{(n)} = \frac{1}{2}\int_{\mathbb{C}^2}|\chi_A(z)-\chi_A(w)||\mathscr{K}_n(z,w)|^2 dA(z)dA(w)$$
$$= \int_A\int_{A^c}|\mathscr{K}_n(z,w)|^2 dA(z)dA(w).$$

- Lacroix-A-Chez-Toine, Majumdar and Schehr '19 showed for $Q(z) = |z|^2$ (Ginibre ensemble) that for $A = \{z \in \mathbb{C} : |z| \le a\}$

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \operatorname{Var} N_A^{(n)} = \frac{a}{\sqrt{\pi}} \sqrt{\Delta Q(a)}.$$

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- They also showed what happens in the case of a microscopic dilation of the droplet $S_Q = \overline{\mathbb{D}}$.
Let $A = A_n(\delta) = \{z \in \mathbb{C} : |z| \leq 1 + \frac{\delta}{\sqrt{2n\Delta Q(1)}}\}$ for $\delta \in \mathbb{R}$, then

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \operatorname{Var} N_{A_n(\delta)}^{(n)} = \frac{f(\delta)}{\sqrt{\pi}}, \quad f(\delta) = \sqrt{2\pi} \int_\delta^\infty \frac{\operatorname{erfc}(t)\operatorname{erfc}(-t)}{4} dt$$

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- This was shown to be **universal** in the rotational symmetric case by Akemann, Byun, Ebke '23. Assumptions: $Q(z) = g(|z|)$ where $(rg'(r))' > 0$, $g'(1) = 2$ and $rg'(r) \to 0$ as $r \downarrow 0$.

- It was proved by Lin '24 and by Levi, Marzo and Ortega-Cerdà '24 that for the Ginibre ensemble $Q(z) = |z|^2$ one has

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \operatorname{Var} N_A^{(n)} = \frac{1}{2\pi\sqrt{\pi}} \int_{\partial_* A} d\mathscr{H}^1(z) = \frac{1}{2\pi\sqrt{\pi}} \mathscr{H}^1(\partial_* A)$$

for any Borel set $A \subset \mathbb{C}$, where $d\mathscr{H}^1(z)$ is the one-dimensional Hausdorff measure on the measure theoretic boundary $\partial_* A$.

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- The measure theoretic boundary is defined explicitly by $\partial_* A =$

$$\left(\left\{ z \in \mathbb{C} : \lim_{r\downarrow 0} \frac{\lambda_2(A \cap B(z,r))}{\pi r^2} = 1 \right\} \cup \left\{ z \in \mathbb{C} : \lim_{r\downarrow 0} \frac{\lambda_2(A \cap B(z,r))}{\pi r^2} = 0 \right\} \right)^c$$

When A has a C^1 boundary, $\partial_* A$ is the same as the topological boundary ∂A, and $d\mathscr{H}^1(\partial_* A)$ is just the usual arc length differential.

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- Heuristically, one can roughly argue the result from the peaked behavior of $|K_n(z,w)| \approx n\Delta Q(z) e^{-n\Delta Q(z)|z-w|^2}$ around the diagonal $z = w$ (while in the bulk $\mathring{S}_Q$) and the formula

$$\operatorname{Var} N_A^{(n)} = \int_A \int_{A^c} |\mathscr{K}_n(z,w)|^2 dA(z) dA(w).$$

Theorem (Akemann - Duits - M. '24)

Consider a random normal matrix model with a C^2 potential Q which is assumed to be real analytic in a neighborhood of S_Q. Fix a compact set $K \subset \mathring{S}_Q$ and assume that $\Delta Q > 0$ on K. Then we have

$$\operatorname{Var} N_A^{(n)} \asymp \sqrt{n}|\partial A|$$

as $n \to \infty$ for all convex sets $A \subset K$ with a C^2 boundary, where the implied constants depend only on Q and K.
When ΔQ is constant on K we have for such sets A that as $n \to \infty$

$$\operatorname{Var} N_A^{(n)} = \frac{\sqrt{n}}{2\pi\sqrt{\pi}}|\partial A|\sqrt{\Delta Q|_K} + \mathscr{O}(1).$$

Theorem (Akemann - Duits - M. '24)

(elliptic Ginibre ensemble) Consider $Q(z) = (|z|^2 - \tau \operatorname{Re}(z^2))/(1-\tau^2)$ *with fixed* $0 \leq \tau < 1$ *and let* $\vec{n}(z)$ *denote the outward unit normal vector at* z *on* ∂S_Q. *Define*

$$A = A_n(S) = \begin{cases} S_Q \cup \left\{[z, z + \frac{\delta}{\sqrt{2n\Delta Q(z)}}\vec{n}(z)] : z \in \partial S_Q\right\}, & \delta \geq 0, \\ S_Q \setminus \left\{[z + \frac{\delta}{\sqrt{2n\Delta Q(z)}}\vec{n}(z), z] : z \in \partial S_Q\right\}, & \delta < 0. \end{cases}$$

Then we have

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \operatorname{Var} N_A^{(n)} = \frac{1}{2\pi\sqrt{\pi}} f(\delta) |\partial A| \sqrt{\Delta Q(z)},$$

$$f(\delta) = \sqrt{2\pi} \int_\delta^\infty \frac{\operatorname{erfc}(t)\operatorname{erfc}(-t)}{4} dt.$$

$$S_Q = \left\{z \in \mathbb{C} : \left(\frac{\operatorname{Re} z}{1+\tau}\right)^2 + \left(\frac{\operatorname{Im} z}{1-\tau}\right)^2 \leq 1\right\}$$

Bulk Theorem (Marzo - M. - Ortega-Cerdà '25)

Consider a random normal matrix model with a potential Q that is C^2, real analytic on $\mathring{S}_Q$ and $\Delta Q > 0$ on S_Q. For any Borel set $A \Subset \mathring{S}_Q$ we have

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \operatorname{Var} N_A^{(n)} = \frac{1}{2\pi\sqrt{\pi}} \int_{\partial_* A} \sqrt{\Delta Q(z)} d\mathscr{H}^1(z),$$

where $\partial_* A$ is the measure theoretic boundary of A.

- For any $\delta \in \mathbb{R}$, we define the following tubular neighborhood of ∂S_Q

$$S_{Q,n}^{\delta} = \{h_n(z,t) : z \in \partial S_Q,\ |t| < |\delta|\},$$

where

$$h_n(z,t) = z + \frac{1}{\sqrt{2n\Delta Q(z)}}\vec{n}(z)t.$$

Here $\vec{n}(z)$ denotes the outward unit normal vector on ∂S_Q at z. Now consider our counting statistic $N_A^{(n)}$ for

$$A = A_n(\delta) = \begin{cases} S_Q \cup S_{Q,n}^{\delta}, & \delta \geq 0, \\ S_Q \setminus S_{Q,n}^{\delta}, & \delta < 0. \end{cases}$$

Edge Theorem (Marzo - M. - Ortega-Cerdà '25)

Consider a random normal matrix model with a potential Q that is C^2 on $\mathbb{C}$, and real analytic and strictly subharmonic on a neighborhood of S_Q. Assume that S_Q is simply connected and that it has a smooth boundary. Then

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \operatorname{Var} N_{A_n(\delta)}^{(n)} = \frac{f(\delta)}{2\pi\sqrt{\pi}} \int_{\partial S_Q} \sqrt{\Delta Q(z)}\, d\omega_{S_Q^c}^{\infty}(z)$$

uniformly for $\delta \in \mathbb{R}$ in compact sets, where $\omega_{S_Q^c}^{\infty}$ is the harmonic measure at ∞, and

$$f(\delta) = \sqrt{2\pi} \int_{\delta}^{\infty} \frac{\operatorname{erfc}(t)\operatorname{erfc}(-t)}{4} dt.$$

The harmonic measure at ∞ corresponding to S_Q^c is given by

$$d\omega_{S_Q^c}^{\infty}(z) = |\phi'(z)| d\mathscr{H}^1(z),$$

where ϕ is any conformal map from S_Q^c to $\overline{\mathbb{D}}^c$ satisfying $\phi(\infty) = \infty$.

How to prove the Bulk Theorem?

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$$BV(\mathbb{C}) = \{f \in L^1(\mathbb{C}) : [f]_{BV} < \infty\},$$

where $[f]_{BV}$ denotes the total variation of f:

$$[f]_{BV} = \sup\left\{\int_{\mathbb{C}} f(z)\,\mathrm{div}\,\phi(z)\,dA(z) : \phi \in C_c^\infty(\mathbb{C}, \mathbb{R}^2) \text{ with } \|\phi\|_{L^\infty} \leq 1\right\}.$$

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- Any $f \in BV(\mathbb{C})$ can be approximated by functions in $C_c^\infty(\mathbb{C})$ in the following way. There exists a sequence $(f_j)_{j=1}^\infty$ in $C_c^\infty(\mathbb{C})$ such that

$$\lim_{j\to\infty} \|f - f_j\|_{L^1} = 0 \text{ and } \lim_{j\to\infty} \int_{\mathbb{C}} |\nabla f_j(z)|\,dA(z) = [f]_{BV}.$$

(In general, it is not possible to approximate a BV function f by a sequence $f_j \in C_c^\infty(\mathbb{C})$ in $W^{1,1}$ norm.)

- It turns out that (structure theorem of De Giorgi 1955)

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- The strategy is to use the sequence $(f_j)_{j=1}^{\infty}$ to approximate the variance

$$\begin{aligned}\operatorname{Var} N_A^{(n)} &= \frac{1}{2}\int_{\mathbb{C}^2} |\chi_A(z) - \chi_A(w)||\mathscr{K}_n(z,w)|^2 dA(z)dA(w) \\ &\approx \frac{1}{2}\int_{\mathbb{C}^2} |f_j(z) - f_j(w)||\mathscr{K}_n(z,w)|^2 dA(z)dA(w) \\ &\approx \frac{1}{2\pi\sqrt{\pi}}\|\sqrt{\Delta Q}\nabla f_j\|_{L^1},\end{aligned}$$

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for large n, and then take $j \to \infty$.
- See Le Doussal and Schehr '25 for the rotational symmetric case.

How to prove the Edge Theorem?

- The number variance is essentially dictated by the boundary ∂S_Q.

$$\operatorname{Var} N_A^{(n)} = \int_A \int_{A^c} |\mathscr{K}_n(z,w)|^2 dA(z) dA(w)$$

$$\approx \int_{\partial S_Q} \int_{\substack{\partial S_Q, \\ d(z_0,w_0) \leq \sqrt{\frac{\log n}{n}}}} \int_{-\epsilon_n}^{\delta} \int_{\delta}^{\epsilon_n} \frac{|\phi'(z_0)||\phi'(w_0)|}{n\sqrt{\Delta Q(z_0) \Delta Q(w_0)}} \times$$

$$\left| \mathscr{K}_n \left(z_0 + \frac{\vec{n}(z_0)\xi}{\sqrt{n \Delta Q(z_0)}}, w_0 + \frac{\vec{n}(w_0)\eta}{\sqrt{n\Delta Q(w_0)}} \right) \right|^2 d\xi d\eta d\mathscr{H}^1(w_0) d\mathscr{H}^1(w_0).$$

where $d(z_0, w_0) = |\log(\phi(z_0)\overline{\phi(w_0)})|$ and $\epsilon_n = \text{const.} \times \sqrt{\log n}$.

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- We have to understand the behavior of $|\mathscr{K}_n(z,w)|$ when z and w are in the vicinity of the boundary (order $\mathscr{O}(\sqrt{\frac{\log n}{n}})$ from ∂S_Q).

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where $d(z_0,w_0) = |\log(\phi(z_0)\overline{\phi(w_0)})|$ and $\epsilon_n = \text{const.} \times \sqrt{\log n}$.

- We have to understand the behavior of $|\mathscr{K}_n(z,w)|$ when z and w are in the vicinity of the boundary (order $\mathscr{O}(\sqrt{\frac{\log n}{n}})$ from ∂S_Q).
- A distinction has to be made between z_0 and w_0 close to each other ($z_0 - w_0 = \mathscr{O}(\sqrt{\frac{\log n}{n}})$) and the case where they are not.

Lemma (Marzo - M. - Ortega-Cerdà '25)

Assume the conditions on Q from the Edge Theorem.
Let $z_0, w_0 \in \partial S_Q$ and denote by $\vec{n}(z_0)$ and $\vec{n}(w_0)$ the outward unit normal vectors on ∂S_Q at z_0 and w_0. Then we have as $n \to \infty$ that

$$\frac{1}{n\sqrt{\Delta Q(z_0)\Delta Q(w_0)}}\left|\mathscr{K}_n\left(z_0+\frac{\vec{n}(z_0)\xi}{\sqrt{n\Delta Q(z_0)}}, w_0+\frac{\vec{n}(w_0)\eta}{\sqrt{n\Delta Q(w_0)}}\right)\right| =$$

$$(\frac{1}{2}+\mathscr{O}(\frac{\log^3 n}{\sqrt{n}}))\exp\left(-\frac{1}{2}|\xi-\eta|^2 - n\Delta Q(z_0)\frac{(\log(\phi(z_0)\overline{\phi(w_0)}))^2}{2|\phi'(z_0)|^2}\right)$$

$$\left|\operatorname{erfc}\left(\frac{\xi+\overline{\eta}}{\sqrt{2}}+\sqrt{n\Delta Q(z_0)}\frac{\log(\phi(z_0)\overline{\phi(w_0)})}{\sqrt{2}|\phi'(z_0)|}\right)\right|$$

uniformly for $|z_0-w_0| = \mathscr{O}(\sqrt{\frac{\log n}{n}})$ and $\xi, \eta = \mathscr{O}(\sqrt{\log n})$, where ϕ is the conformal map from S_Q^c to $\overline{\mathbb{D}}^c$ such that $\phi(\infty)=\infty$ and $\phi'(\infty)>0$.

Generalizes Hedenmalm-Wennman '21 where $z_0 = w_0$ and $\xi, \eta = \mathscr{O}(1)$.

Lemma (Marzo - M. - Ortega-Cerdà '25)

Consider a random normal matrix model with a potential Q that is C^2 on $\mathbb{C}$, and real analytic and strictly subharmonic on a neighborhood of S_Q. Assume S_Q is simply connected and has a smooth boundary. As $n \to \infty$

$$\frac{1}{\sqrt{n}} \left| \mathscr{K}_n \left(z_0 + \frac{\vec{n}(z_0)\xi}{\sqrt{n\Delta Q(z_0)}}, w_0 + \frac{\vec{n}(w_0)\eta}{\sqrt{n\Delta Q(w_0)}} \right) \right| \leq C_Q \left| \mathscr{S} \left(z_0 + \frac{\vec{n}(z_0)\xi}{\sqrt{n\Delta Q(z_0)}}, w_0 + \frac{\vec{n}(w_0)\eta}{\sqrt{n\Delta Q(w_0)}} \right) \right| e^{-(\operatorname{Re}\xi)^2} e^{-(\operatorname{Re}\eta)^2},$$

uniformly for $z_0, w_0 \in \partial S_Q$ and $\xi, \eta = \mathscr{O}(\sqrt{\log n})$, where $C_Q > 0$, and $\mathscr{S}$ is the Szegő kernel associated with ∂S_Q.

$$\mathscr{S}(z,w) = \frac{1}{2\pi} \frac{\sqrt{\phi'(z)}\overline{\sqrt{\phi'(w)}}}{\phi(z)\overline{\phi(w)} - 1},$$

Related result by Ameur and Cronvall '23 with condition $|z_0 - w_0| \geq \epsilon$.

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HAPPY BIRTHDAY PETER!

THANK YOU!