Real Infinitesimal Freeness

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limit distributions of random matrices

- $\{A_N\}_N$ is an ensemble of random matrices, $\mathrm{Tr} =$ un-normalized trace and $\mathrm{tr} = N^{-1}\mathrm{Tr} =$ normalized trace
- for a polynomial p, $\mu_N(p) = \mathrm{E}(\mathrm{tr}(p(A_N)))$
- if μ_N converges pointwise on polynomials to μ, we say the ensemble $\{A_N\}_N$ has the *limit distribution* μ
- let $\mu_N' = N(\mu_N - \mu)$, μ_N' is linear on polynomials and $\mu_N'(1) = 0$
- we say the pair (μ_N, μ_N') is an *infinitesimal law* on $\mathcal{A} = \mathbf{C}[t]$ (polynomials in t) and the triple $(\mathcal{A}, \mu_N, \mu_N')$ is an (*infinitesimal probability space*)
- if μ_N' converges pointwise to μ' we have that (μ, μ') is an infinitesimal law on $\mathcal{A}$ and we say that $\{A_N\}_N$ has the *infinitesimal limit distribution* (μ, μ')

general rules, valid in many examples

- $\mu_N(p) = \mathrm{E}(\mathrm{tr}(p(A_N)))$
- we imagine: $\mu_N(p) = \mu(p) + \frac{\mu'(p)}{1!\,N} + \frac{\mu''(p)}{2!\,N^2} + \frac{\mu'''(p)}{3!\,N^3} + \cdots$

This means

- if $\mu := \lim_N \mu_N$ exists, we set $\mu'_N = N(\mu_N - \mu)$, or equivalently $$\mu_N = \mu + \frac{\mu'_N}{N}$$
- if $\mu' := \lim_N \mu'_N$ exists, we set $\mu''_N = 2N(\mu'_N - \mu')$, or equivalently $\mu_N = \mu + \frac{\mu'}{N} + \frac{\mu''_N}{2N^2}$

We write in general, *assuming* all these limits exist

- $\mu_N^{(k+1)} = (k+1)N(\mu_N^{(k)} - \mu^{(k)})$ and
- $\mu^{(k+1)} := \lim_N \mu_N^{(k+1)}$

examples (Johansson (1998), Dumitriu & Edelman (2006))

- $G = (g_{ij})_{i,j=1}^{N}$ with $\{g_{ij}\}_{i,j}$ real independent $\mathcal{N}(0,1)$
- $A_N = \frac{1}{\sqrt{2N}}(G + G^t) = N \times N$ *Gaussian Orthogonal Ensemble*
- it has the limit infinitesimal distribution (μ, μ') where μ is Wigner's semi-circle law and $\mu' = \frac{1}{2}(\nu_1 - \nu_2)$ is a signed measure with $\nu_1 = \frac{1}{2}(\delta_{-2} + \delta_2)$ (Dirac masses at ± 2) and $d\nu_2(t) = \frac{1}{\pi}\frac{1}{\sqrt{4-t^2}}$ (arcsine law on $[-2, 2]$)

- $G = (g_{ij})_{\substack{i=1,\dots N \\ j=1,\dots,M}}$ $(N \times M)$, $\{g_{ij}\}_{i,j}$ real independent $\mathcal{N}(0,1)$
- $W_N = \frac{1}{N}GG^t$, $N \times N$ *real Wishart matrix*
- if $\frac{M}{N} \longrightarrow c$ then $\{W_N\}_N$ has the limit distribution $\mu =$ the Marchenko-Pastur Law with parameter c
- if $N(\frac{M}{N} - c) \longrightarrow c'$ then $\{W_N\}_N$ has the infinitesimal limit distribution $\mu' =$ with parameter c'

Infinitesimal Marchenko-Pastur (details)

- μ_c (where $M/N \to c > 0$)
- $a = (1 - \sqrt{c})^2$, and $b = (1 + \sqrt{2})^2$
- $d\mu_c(t) = (1-c)\delta_0 + \frac{\sqrt{(b-t)(t-a)}}{2\pi t}\, dt$
- $\mu_c' = \frac{1}{2}(\nu_1 - \nu_2) - c'(\rho_1 - \rho_2)$
- $\nu_1 = \frac{1}{2}(\delta_a + \delta_b)$, $d\nu_2(t) = \frac{dt}{\pi\sqrt{(b-t)(t-a)}}$ on $[a, b]$
- $\rho_1 = \delta_0$ and $\rho_2 = \frac{t+1-c}{2\pi t\sqrt{(b-t)(t-a)}}$ on $[a, b]$
- ρ_1 is absent when $c > 1$

free independence, formalized by Voiculescu (1983)

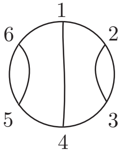
- many random matrix ensembles exhibit asymptotic freeness, to formalize
- we have a unital algebra $\mathcal{A}$, the random variables of our model
- $\varphi : \mathcal{A} \to \mathbf{C}$, our expectation, $\varphi(1) = 1$
- free independence gives a universal rule for computing moments: when $a, b \in \mathcal{A}$ are *free* with respect to φ we have
- $\varphi(abab) = \varphi(a^2)\varphi(b)^2 + \varphi(a)^2\varphi(b^2) - \varphi(a)^2\varphi(b)^2$
- rule becomes easier if elements are *centred*: $\varphi(a) = \varphi(b) = 0$, then the rule is $\varphi(abab) = 0$
- *in general*: subalgebras $\mathcal{A}_1, \ldots, \mathcal{A}_s$ are φ-freely independent if $\varphi(a_1 \cdots a_n) = 0$, when ever each a_i is centred and $a_1, \ldots, a_n$ is *alternating*: $a_i \in \mathcal{A}_{j_i}$ and $j_1 \neq j_2 \neq \cdots \neq j_n$

free cumulants, Speicher (1994)

- Voiculescu's rule is hard to use with random matrix ensembles because it requires centering
- freeness is equivalent to the vanishing of mixed cumulants
- free cumulants are a family $\{\kappa_n\}_n$ of multi-linear maps: $\kappa_n : \mathcal{A}^{\otimes n} \to \mathbf{C}$
- follows the construction of Thiele (1898), later systemized by Rota (1963): $\kappa_1(a) = \varphi(a)$,

  $\kappa_2(a_1, a_2) = \varphi(a_1 a_2) - \varphi(a_1)\varphi(a_2)$,

  $\kappa_3(a_1, a_2, a_3) = \varphi(a_1 a_2 a_3) - \varphi(a_1)\varphi(a_2 a_3) - \varphi(a_2)\varphi(a_1 a_3) - \varphi(a_1 a_2)\varphi(a_3) + 2\varphi(a_1)\varphi(a_2)\varphi(a_3)$
- the general rule uses non-crossing partitions:

$NC(n) = \Bigg\{$ 

$\Bigg\} : \left\{ \begin{array}{l} \text{in general we partition the} \\ \text{set } [n] \text{ into disjoint blocks} \\ \text{so the blocks don't cross} \end{array} \right\}$

infinitesimal free cumulants, Fevrier & Nica (2009)

- $$\kappa_{\sqcup\sqcup\shortmid}(a_1, a_2, a_3, a_4, a_5) = \kappa_2(a_1, a_2)\kappa_2(a_3, a_4)\kappa_1(a_5)$$
- $\varphi(x_1 \cdots x_n) = \sum_{\pi \in NC(n)} \kappa_\pi(x_1, \ldots, x_n)$
- $\boxed{\varphi'(x_1 \cdots x_n) = \sum_{\pi \in NC(n)} \partial\kappa_\pi(x_1, \ldots, x_n)}$
- where $\partial\kappa_\pi(x_1 \cdots x_n)$
  $= \sum_{V \in \pi} \kappa'_{|V|}(x_1, \ldots, x_n \mid V) \prod_{W \neq V} \kappa_{|W|}(x_1, \ldots, x_n \mid W)$
- i.e. $\partial\kappa_\pi$ is defined by the Leibniz rule, for example when $\pi = \{(1,3), (2), (4)\}$

$$\begin{aligned}\partial\kappa_\pi(x_1, x_2, x_3, x_4) &= \partial(\kappa_2(x_1, x_3)\kappa_1(x_2)\kappa_1(x_4)) \\ &= \kappa'_2(x_1, x_3)\kappa_1(x_2)\kappa_1(x_4) + \kappa_2(x_1, x_3)\kappa'_1(x_2)\kappa_1(x_4) \\ &\qquad + \kappa_2(x_1, x_3)\kappa_1(x_2)\kappa'_1(x_4)\end{aligned}$$

applications to Gaussian and Wishart matrices

in the limit as $N \to \infty$

- for GUE, $\kappa_n = \begin{cases} 1 & n = 2 \\ 0 & \text{otherwise} \end{cases}$, $\kappa_n'^{(c)} = 0$
- for GOE, $\kappa_n = \begin{cases} 1 & n = 2 \\ 0 & \text{otherwise} \end{cases}$, $\kappa_n'^{(c)} = \begin{cases} 1 & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$
- for complex Wishart $\kappa_n = c$, $\kappa_n'^{(c)} = c'$
- for real Wishart $\kappa_n = c$, $\kappa_n'^{(c)} = c' + a_n$
-

n	1	2	3	4	5
a_n	0	$-c$	$-3c$	$-(6c + 3c^2)$	$-(10c + 5c^2)$

- independent GOE and real Wishart are not asymptotically free, but complex ones are

comparison of unitary and orthogonal invariance

$$E(\mathrm{Tr}(UA_1U^{-1}A_2)) = N^{-1}\mathrm{Tr}(A_1)\mathrm{Tr}(A_2) \text{ and}$$
$$E(\mathrm{Tr}(UA_1UA_2)) = 0.$$

$$E(\mathrm{Tr}(OA_1O^{-1}A_2)) = N^{-1}\mathrm{Tr}(A_1)\mathrm{Tr}(A_2) \text{ and}$$
$$E(\mathrm{Tr}(OA_1OA_2)) = N^{-1}\mathrm{Tr}(A_1A_2^t).$$

- se we need a **new** theory for orthogonal invariance
- the fourth example shows that **transposes** must be incorporated into the theory

integration by parts on the orthogonal group

- $O \in \mathcal{O}_N$ = $N \times N$ orthogonal matrices with Haar measure
- $K_{ij} = E_{ij} - E_{ji}$, $\{E_{ij}\}_{ij}$ matrix units form $M_N(\mathbf{C})$
- $\partial_{ij} =$ derivative in the direction of K_{ij}, $\Delta = \sum_{i<j} \partial^2_{K_{ij}}$
- f and g twice differentiable from $\mathcal{O}_N \to M_N(\mathbf{C})$
- $\Gamma(f \otimes g) := \sum_{i<j} \partial_{ij} f \otimes \partial_{ij}\, g$ (*carré de champs* operator)
- $\Delta(f \otimes g) = \Delta(f) \otimes g + 2\Gamma(f \otimes g) + f \otimes \Delta(g)$
- $\mathrm{E}(\Gamma(f \otimes g)) = -\mathrm{E}(\Delta(f) \otimes g)$ (*expectation taken entrywise*)
- $F(A \otimes B) := \mathrm{Tr}(AB)$
- $$\boxed{\mathrm{E}(F(\Gamma(f \otimes g))) = -\mathrm{E}(F(\Delta(f) \otimes g))}$$

integration by parts $\Rightarrow$ integration formula

- $M_1, \ldots, M_{2n}$ independent from O, Tr = un-normalized trace

$$
\begin{aligned}
&(N-1)\cdot \mathrm{E}\big[\mathrm{Tr}(OM_1O^t \cdot M_2 \cdot OM_3O^t \cdots OM_{n-1}O^t \cdot M_n)\big] \\
&= -\sum_{\substack{k=1\\ k \text{ odd}}}^{n-1} \mathrm{E}\big[\mathrm{Tr}(OM_1O^t \cdots OM_kO^t \cdot (M_{k+1} \cdots OM_{n-1}O^t \cdot M_n)^t)\big] \\
&+ \sum_{\substack{k=1\\ k \text{ odd}}}^{n-1} \mathrm{E}\big[\mathrm{Tr}(OM_1O^t \cdots OM_kO^t)\cdot \mathrm{Tr}(M_{k+1} \cdots OM_{n-1}O^t \cdot M_n)\big] \\
&+ \sum_{\substack{k=3\\ k \text{ odd}}}^{n-1} \mathrm{E}\big[\mathrm{Tr}(OM_1O^t \cdots M_{k-1} \cdot (OM_kO^t \cdots OM_{n-1}O^t \cdot M_n)^t)\big] \\
&- \sum_{\substack{k=3\\ k \text{ odd}}}^{n-1} \mathrm{E}\big[\mathrm{Tr}(OM_1O^t \cdots M_{k-1})\cdot \mathrm{Tr}(OM_kO^t \cdots OM_{n-1}O^t \cdot M_n)\big]
\end{aligned}
$$

- $\mathcal{A}_{1,N}, \dots, \mathcal{A}_{s,N} \subseteq M_N(\mathcal{L}^{\infty-})$ be unital subalgebras such that the entries of matrices from different subalgebras form independent sets. Suppose that all, or all but one, of the subalgebras are orthogonally invariant, and suppose that each of the subalgebras satisfies for $P_1, \dots, P_n \in \mathcal{A}_j$

$$\mathrm{E}(\mathrm{tr}(P_i)) = \tau(p_i) + N^{-1}\tau'(p_i) + o(N^{-1}),$$

$$k_2(\mathrm{Tr}(P_1), \mathrm{Tr}(P_2)) \to \tau_2(p_1, p_2) \text{ as } N \to \infty$$

$$N^{r-3}k_r(\mathrm{Tr}(P_1), \dots, \mathrm{Tr}(P_n)) = o(1) \quad \text{for } r \geqslant 3$$

- $P_1, \dots, P_n$ $N \times N$, $P_i \in \mathcal{A}_{j_i}$, cyclically alternating, asymptotically centred, orthogonally invariant

$$\mathrm{E}(\mathrm{Tr}(P_1 \cdots P_r)) = \begin{cases} O(N^{-1}) & n \text{ odd} \\ \tau(p_1 p_{k+1}^t) \cdots \tau(p_k p_r^t) + O(N^{-1}) & n = 2k \end{cases}$$

real infinitesimal freeness

- $\mathcal{A}_1, \dots, \mathcal{A}_s$ unital, symmetric, subalgebras of $(\mathcal{A}, \tau, \tau', t)$, a real tracial infinitesimal probability space are *real infinitesimal free* if

- $\mathcal{A}_1, \dots, \mathcal{A}_s$ are τ-free and

- *whenever* $a, \dots, a_n$ are centred and cyclically alternating we have

- $\tau'(a_1 \cdots a_n) = 0$ for n odd

- $\tau'(a_1 a_2) = 0$

- $\tau'(a_1 \cdots a_n) = \tau(a_1 a_{k+1}^t) \cdots \tau(a_k a_n^t)$ for $n = 2k > 3$

(non-tracial) real infinitesimal freeness

- $\mathcal{A}_1, \ldots, \mathcal{A}_s$ unital, symmetric, subalgebras of $(\mathcal{A}, \tau, \tau', t)$, a real infinitesimal probability space are *real infinitesimal free* if

- $\mathcal{A}_1, \ldots, \mathcal{A}_s$ are τ-free and

- *whenever* $a, \ldots, a_n$ are centred and alternating we have

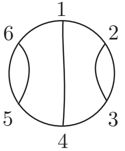
- $\tau'(a_1 a_2) = 0$

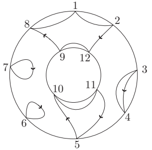
- $\tau'(a_1 \cdots a_n) = \tau(a_1 \tau'(a_2 \cdots a_{n-1}) a_n)$
  $+ \tau(a_1 a_k^t a_n)\tau(a_2 a_{k+1}^t) \cdots \tau(a_{k-1} a_{n-1}^t)$ for $n = 2k - 1$

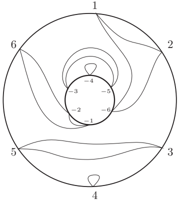
- $\tau'(a_1 \cdots a_n) = \tau(a_1 \tau'(a_2 \cdots a_{n-1}) a_n)$
  $+ \tau(a_1 a_{k+1}^t)\tau(a_2 a_{k+2}^t) \cdots \tau(a_{k1} a_n^t)$ for $n = 2k > 3$

real infinitesimal cumulants

- $\{m_n\}_n$ a moment sequence for a probability measure μ,
- $m_n = \sum_{\pi \in NC(n)} \kappa_\pi$ defines $\{\kappa_n\}_n$ the free cumulants of μ,

where $NC(n) = \left\{ \right.$  $\left. \right\}$

- $\tau_2(x^p, x^q) = \sum_{\pi \in S_{NC}(p,q)} \kappa_\pi$ $\quad S_{NC}(p,q) = \left\{ \right.$  $\left. \right\}$

$S_{NC}^{\delta}(n, -n) = \left\{ \right.$  $\left. \right\}$

in $S_{NC}^{\delta}(n, -n)$ every block appears ‘twice’, the second appearance is a reflection of the first

asymptotically real asymptotic freeness

- $\mathcal{A}_{1,N}, \ldots, \mathcal{A}_{s,N} \subseteq M_N(\mathcal{L}^{\infty-})$ be unital subalgebras such that the entries of matrices from different subalgebras form independent sets. Suppose that all, or all but one, of the subalgebras are orthogonally invariant and that each subalgebra has infinitesimal limite distributions.
- THM The limit distributions are then asymptotically infinitesimally free
- the non-invariant algebra can be constant matrices, in particular finite rank matrices
- EX. if $A_N = P_N X_N P_N$ with $\mathrm{rank}(P_N) = N - k$, with k fixed, then we can find the infinitesimal law of A_N from that of X_N
- the same for $X_N + \theta(I - P_N)$

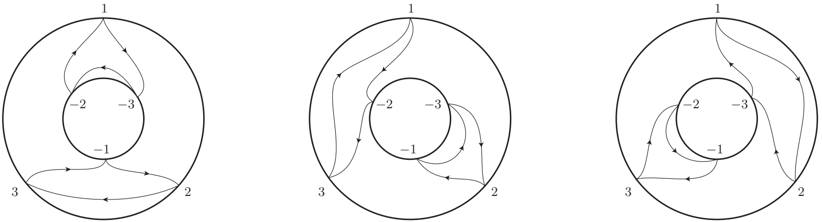
real infinitesimal moment-cumulant formula

- $\tau'(a_1 \cdots a_n) = \sum_{\pi \in NC(n)} (\partial + \delta)\kappa_\pi(a_1, \ldots, a_n)$
- when $\pi = \{(1), (2,4), (3)\}$ we have

$$\partial\kappa_\pi(a_1, a_2, a_3, a_4) = \kappa_1'(a_1)\kappa_2(a_2, a_4)\kappa_1(a_3) + \kappa_1(a_1)\kappa_2'(a_2, a_4)\kappa_1(a_3) + \kappa_1(a_1)\kappa_2(a_2, a_4)\kappa_1'(a_3)$$

$$\delta\kappa_\pi(a_1, a_2, a_3, a_4) = \dot{\kappa}_1(a_1)\kappa_2(a_2, a_4)\kappa_1(a_3) + \kappa_1(a_1)\dot{\kappa}_2(a_2, a_4)\kappa_1(a_3) + \kappa_1(a_1)\kappa_2(a_2, a_4)\dot{\kappa}_1(a_3)$$

spatial derivative



- $S_{NC}^{\delta,a}(n,-n) = \{\sigma \in S_{NC}^{\delta}(n,-n) \mid$ all blocks are through blocks $\}$
- $\dot{\kappa}_n(a_1,\ldots,a_n) = \sum_{\sigma \in S_{NC}^{\delta,a}(n,-n)} \kappa_{\sigma/2}(a_1,\ldots,a_n)$
- $\delta\kappa_\pi(a_1,\ldots,a_n) = \sum_{V\in\pi} \dot{\kappa}_{|V|}(x_1,\ldots,x_n \mid V) \prod_{W\neq V} \kappa_{|W|}(x_1,\ldots,x_n \mid W)$
- $\dot{\kappa}_3(a_1,a_2,a_3) = \kappa_3(a_1,a_3^t,a_2^t) + \kappa_3(a_1,a_2^t,a_3) + \kappa_3(a_1,a_2,a_3^t)$

moment-cumulant & vanishing of mixed cumulants

$$\tau'(a_1 \cdots a_n) \tag{1}$$
$$= \sum_{\pi \in NC(n)} \partial\kappa_\pi(a_1, \ldots, a_n) + \sum_{\sigma \in S_{NC}^{\delta}(n,-n)} \kappa_{\sigma/2}(a_1, \ldots, a_n).$$
$$= \sum_{\pi \in NC(n)} \nabla\kappa_\pi(a_1, \ldots, a_n)$$

$$\nabla\kappa_n(a_1, \ldots, a_n) = \kappa_n'(a_1, \ldots, a_n) + \dot{\kappa}_n(a_1, \ldots, a_n) \tag{2}$$
$$= \sum_{\pi \in NC(n)} \mu(\pi, 1_n)\partial\tau_\pi(a_1, \ldots, a_n).$$

- Let $(\mathcal{A}, \tau, \tau')$ be a real tracial infinitesimal non-commutative probability space and consider unital subalgebras $\mathcal{A}_1, \ldots, \mathcal{A}_s \subset \mathcal{A}$ that are invariant under $a \mapsto a^t$. Then the following statements are equivalent:
  1. The algebras $\mathcal{A}_1, \ldots, \mathcal{A}_s$ are real infinitesimally free.
  2. Mixed free cumulants and mixed infinitesimally free cumulants of the subalgebras vanish.