Discrete and Continuous Muttalib-Borodin process

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Log-gases in caeli australi (AKA Peter b-day)

J. Husson, G. M., A. Occelli: Discrete and Continuous Muttalib--Borodin process: Large deviations and Riemann--Hilbert analysis 2505.23164

TULANE UNIVERSITY

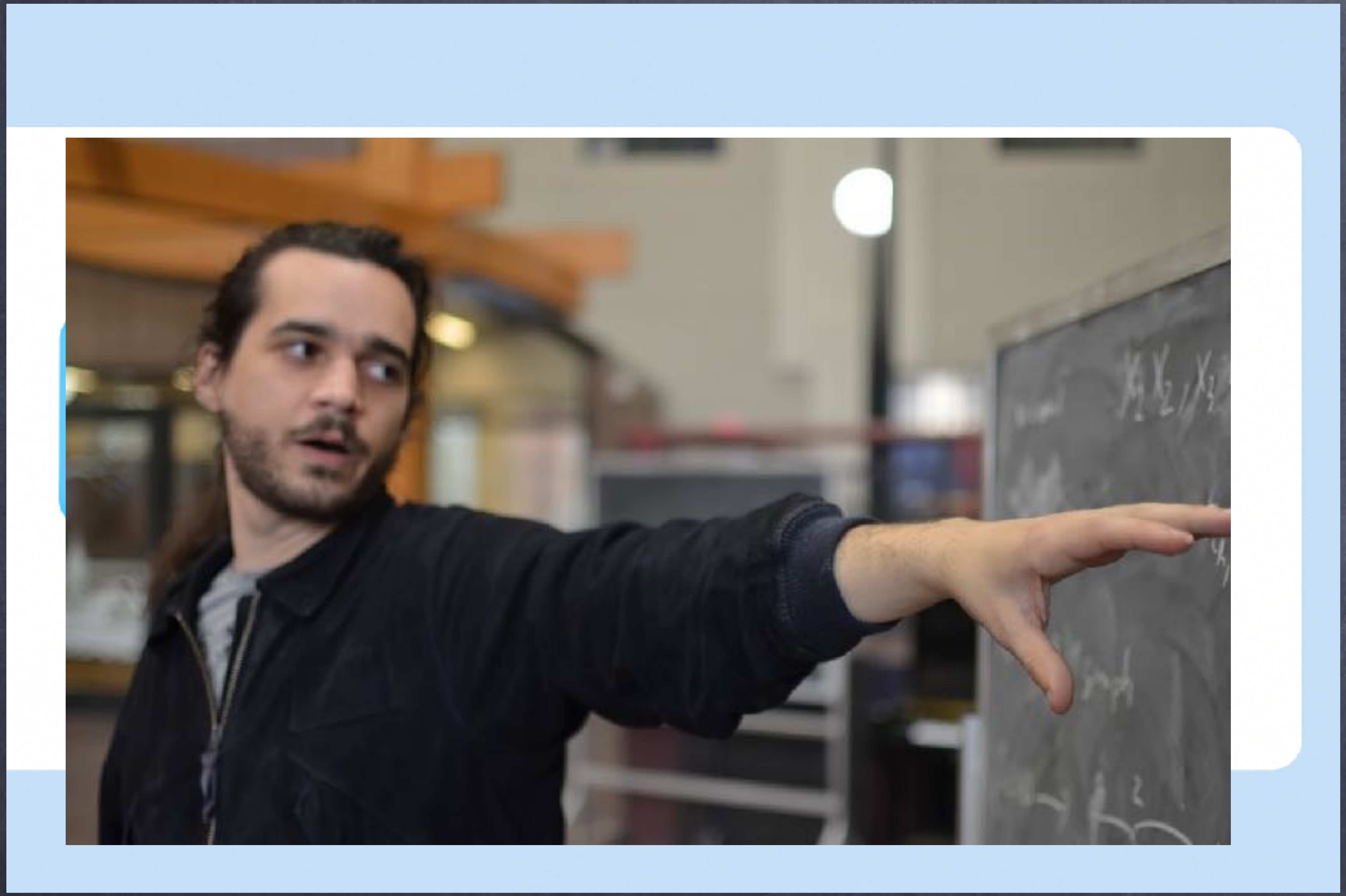
Muttalib-Borodin ensemble

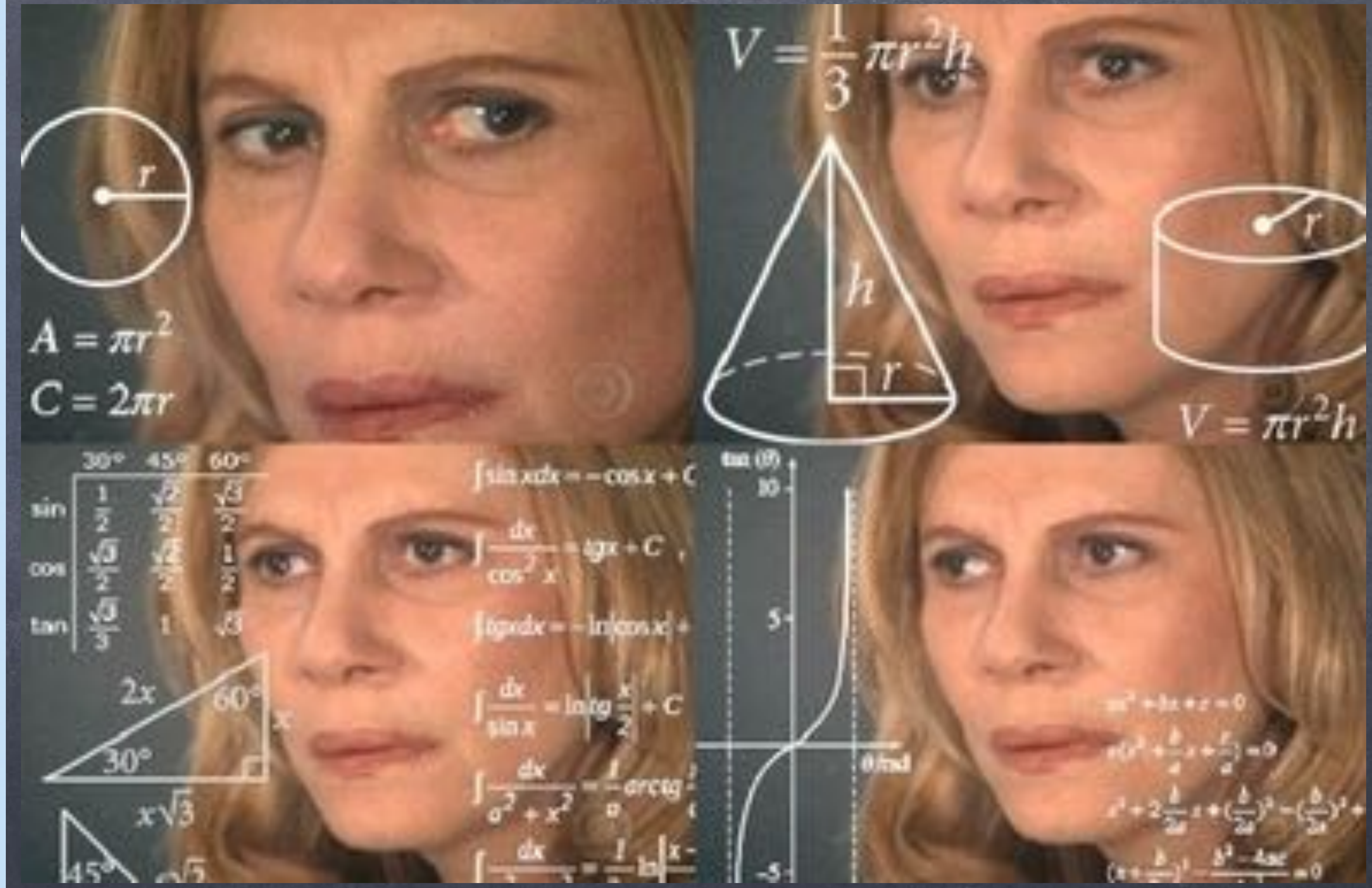
$$d\mathbb{P}(x_1, \ldots, x_N) = \frac{1}{Z_c} \prod_{1 \le i < j \le N} (x_j - x_i)(x_j^\theta - x_i^\theta) \prod_{1 \le i \le N} w_c(x_i) dx_i$$

- Describe statistical properties of disordered systems
- Interacting particle systems
- Example of bi-ortogonal ensemble

See Arno & Don talks

How I met MB

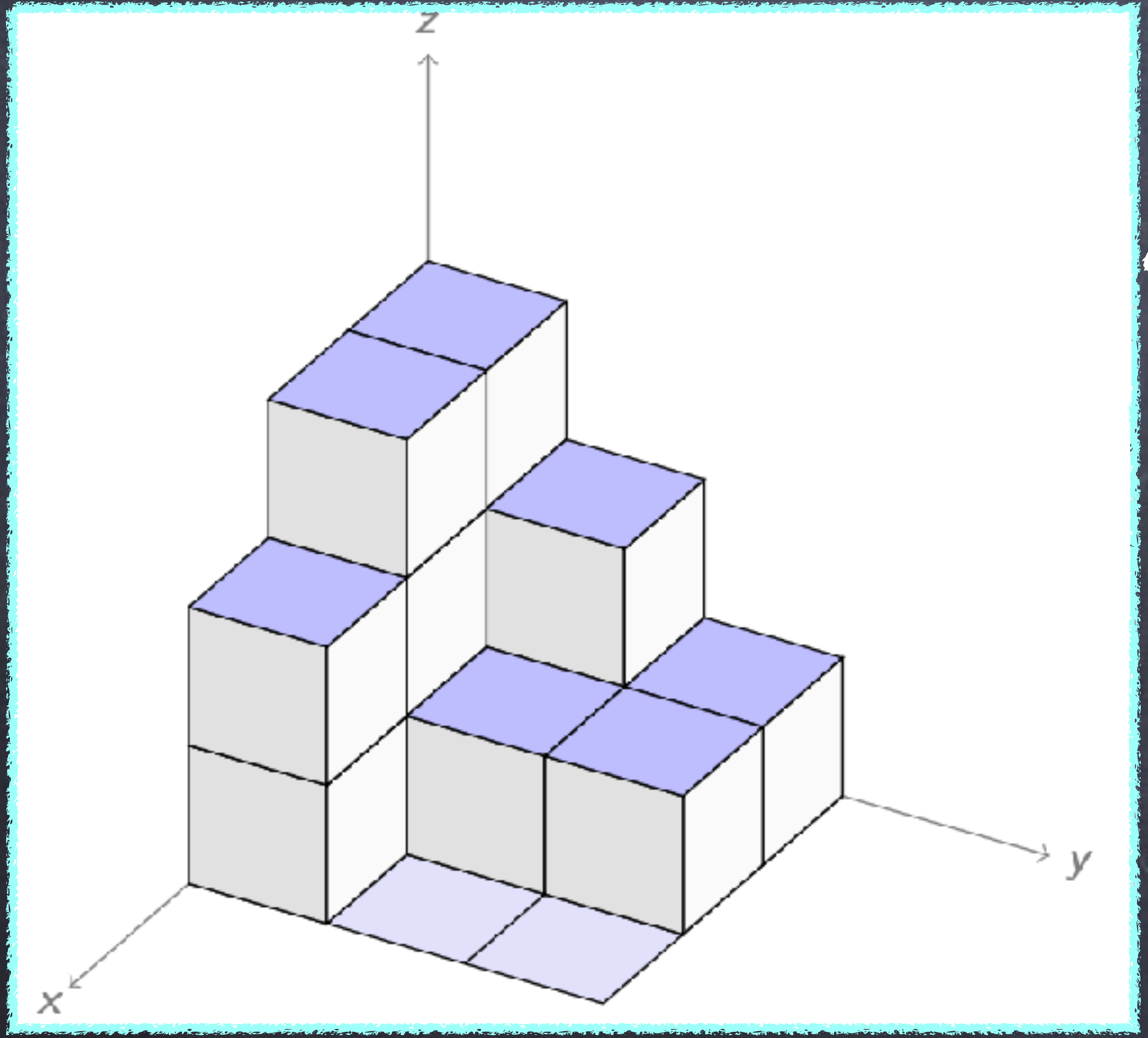


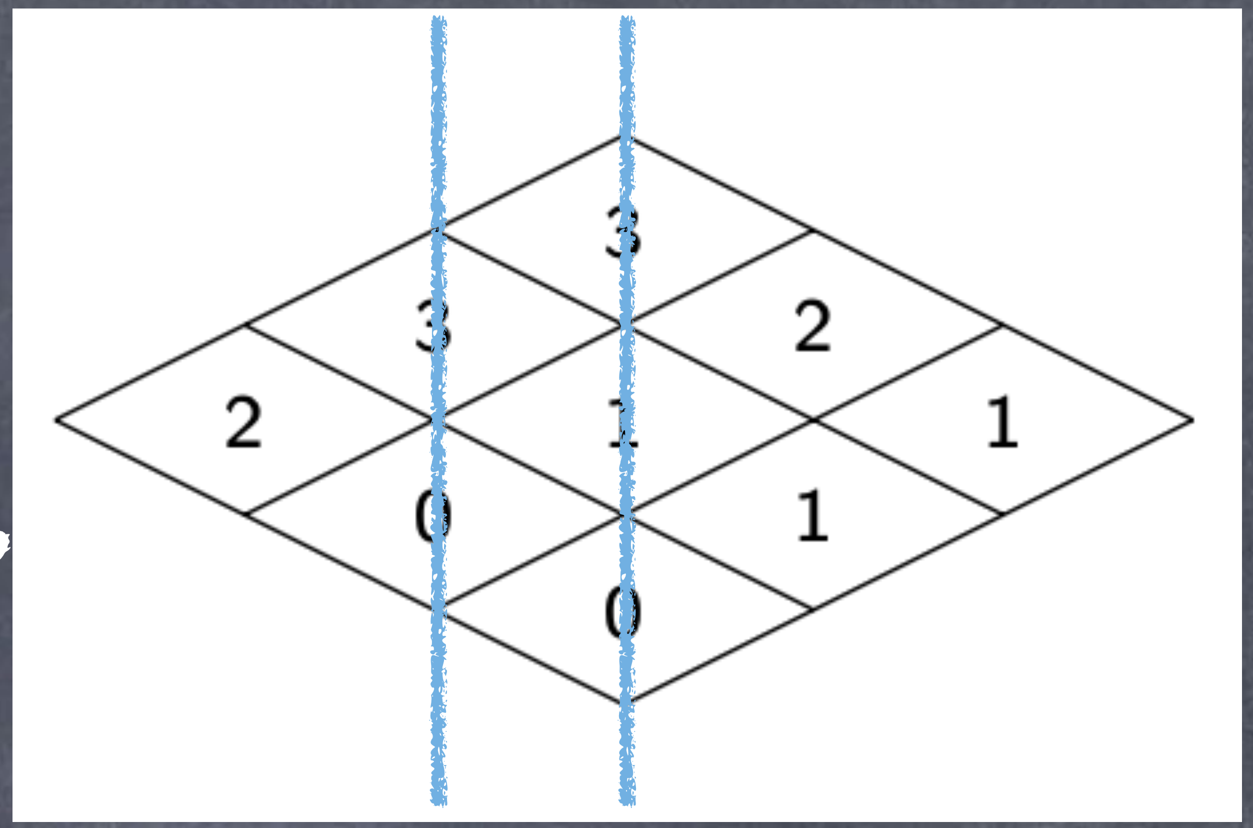


Plane partitions

It is an integer matrix such that

$$\Lambda_{i,j} \geq \max(\Lambda_{i,j+1}, \Lambda_{i+1,j})$$

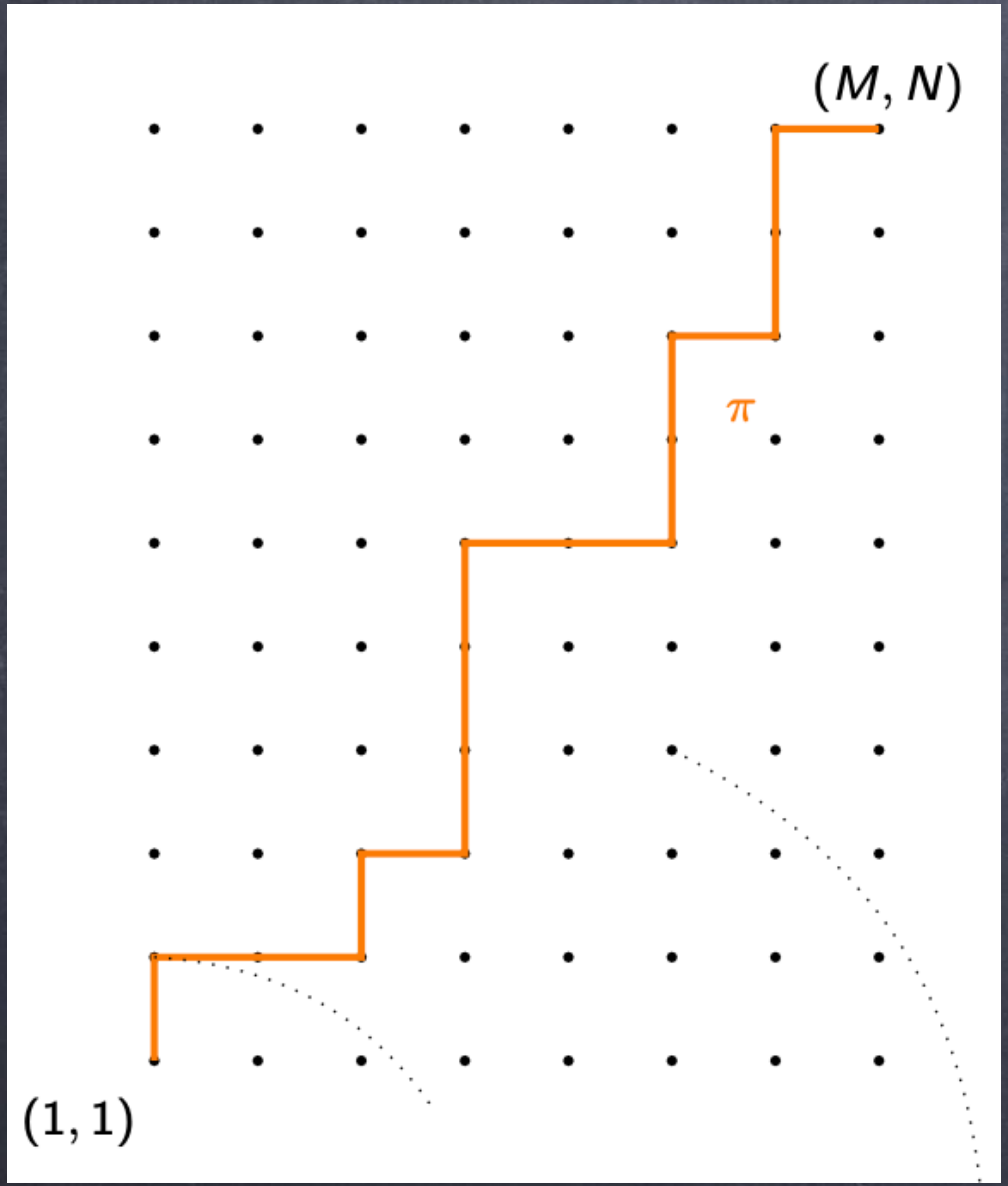




$$3 \geq 3 \geq 1 \geq 0 \geq 0$$

The vertical slice are interlacing partitions

Last passage time

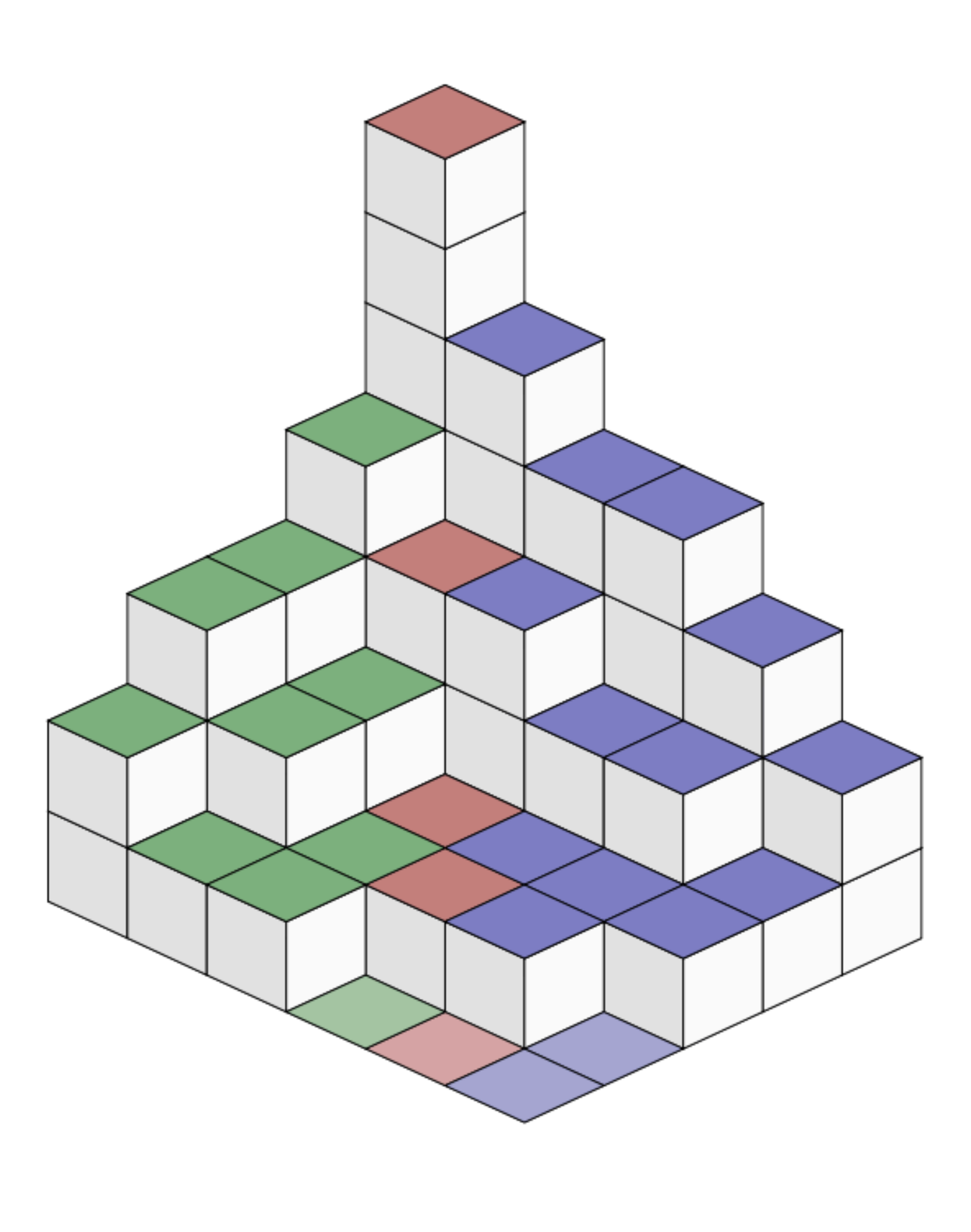


$$\omega_{i,j} \sim \text{Geom}(aq^{\eta(i-1/2)}q^{\theta(j-1/2)})$$

$$L = \max_{\pi:(1,1)\to(M,N)} \sum_{(i,j)\in\pi} \omega_{i,j}$$

Last passage time

RSK algorithm



$\omega_{i,j} \sim \text{Geom}(aq^{\eta(i-1/2)}q^{\theta(j-1/2)})$

$$L = \max_{\pi:(1,1)\to(M,N)} \sum_{(i,j)\in\pi} \omega_{i,j}$$

RSK Occelli, Betea '20,'24

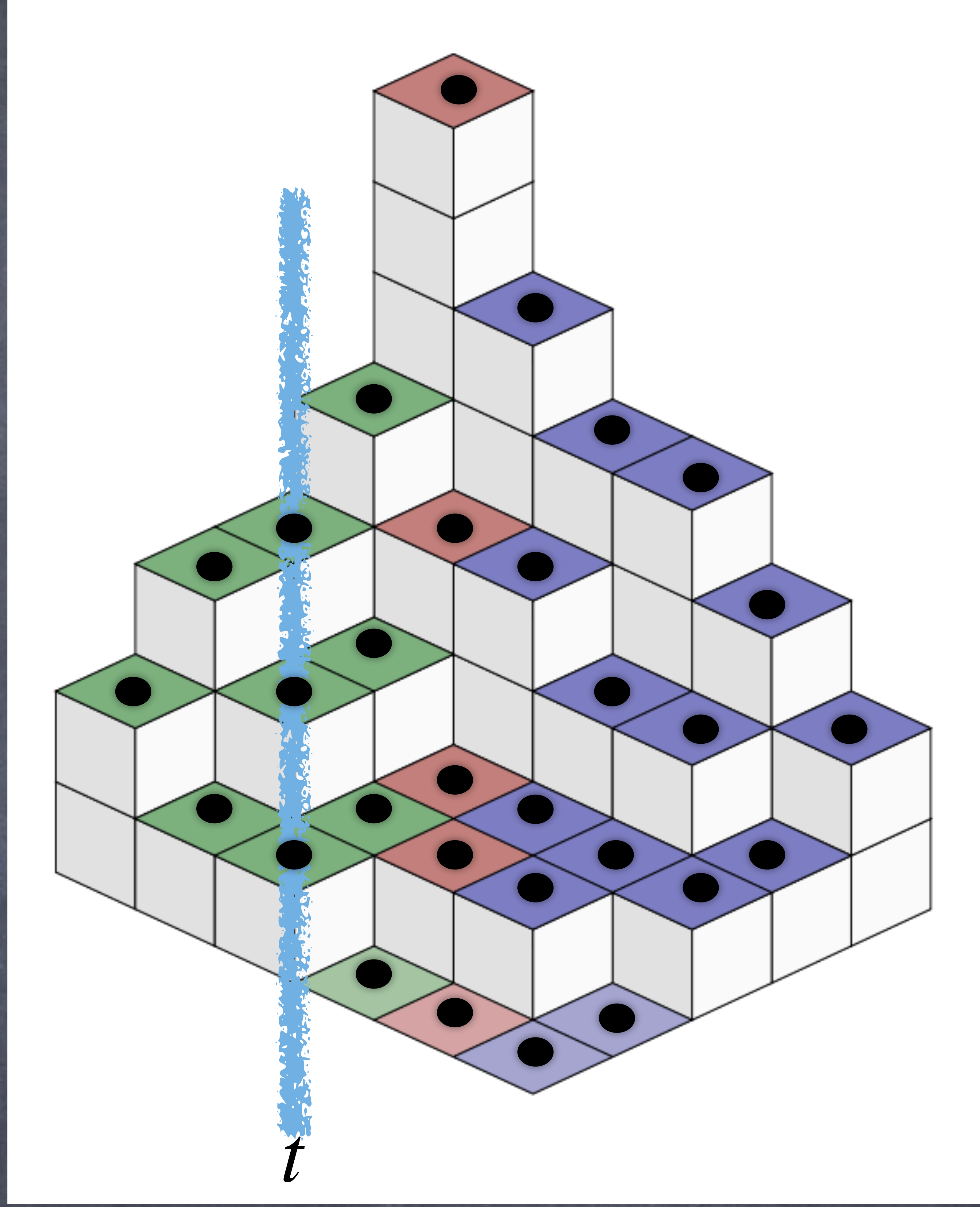
$$\mathbb{P}(\Lambda) \propto q^{\eta LeftVol}(aq^{\frac{\theta+\eta}{2}CentralVol})q^{\theta Rightvol}$$

$$L =_d \Lambda_{1,1}$$

We have a measure on the plane partitions

$$\mathbb{P}(\Lambda) \propto q^{\eta LeftVol}(aq^{\frac{\theta+\eta}{2}CentralVol})q^{\theta Rightvol}$$

Measure on each slice (point process)



$$\mathbb{P}(l^{(t)} = l) = \frac{1}{Z_c} \prod_{1\leq i<j\leq L_t} (q^{\eta l_j} - q^{\eta l_i})(q^{\theta l_j} - q^{\theta l_i}) \prod_{1\leq i\leq L_t} w_d(l_i)$$

Here l_j is the position of the j^{th} particle, L_t is the total length

Scaling Limit

$$\mathbb{P}(l^{(t)} = l) = \frac{1}{Z_c} \prod_{1 \le i < j \le L_t} (q^{\eta l_j} - q^{\eta l_i})(q^{\theta l_j} - q^{\theta l_i}) \prod_{1 \le i \le L_t} w_d(l_i)$$

$$q = e^{-\epsilon}, \qquad a = e^{-\alpha\epsilon}, \qquad x_i^{(t)} = e^{-\epsilon l_i^{(t)}}, \qquad \epsilon \to 0^+$$

$$\mathbb{P}(\boldsymbol{x}^{(t)} = \boldsymbol{x}) dx_1 \ldots dx_{L_t} = \frac{1}{Z_c} \prod_{1 \le i < j \le L_t} (x_j^{\eta} - x_i^{\eta})(x_j^{\theta} - x_i^{\theta}) \prod_{1 \le i \le L_t} w_c(x_i) dx_i$$

$$\mathbb{P}(\boldsymbol{x}^{(t)} = \boldsymbol{x})dx_1 \dots dx_{L_t} = \frac{1}{Z_c} \prod_{1 \le i < j \le L_t} (x_j^\eta - x_i^\eta)(x_j^\theta - x_i^\theta) \prod_{1 \le i \le L_t} w_c(x_i)dx_i$$

Generalization of Muttalib-Borodin:

- Two exponents
- The model comes from a discrete setting
- Setting $\theta = \eta$ we recover the little q-Jacobi polynomials

For $\eta = 1, t = 0, N = M$ one recover the classical Jocobi like MB [Forrester, Wang]

Main object of study

$$\mathbb{P}(\boldsymbol{x}^{(t)} = \boldsymbol{x})dx_1 \dots dx_{L_t} = \frac{1}{Z_c} \prod_{1 \le i < j \le L_t} (x_j^\eta - x_i^\eta)(x_j^\theta - x_i^\theta) \prod_{1 \le i \le L_t} w_c(x_i)dx_i$$

$$\mu_{L_t} = \frac{1}{L_t} \sum_{j=1}^{L_t} \delta_{x_j} \xrightarrow{L_t \to \infty} \mu(dx)$$

Why? $\mu_{L_t} = \frac{1}{L_t}\sum_{j=1}^{L_t} \delta_{x_j}$

- Asymptotic shape of the partition
- Orthogonal polynomials
- Correlation Kernel and related questions

Large Deviation Principle

$$\mathbb{P}(\boldsymbol{x}^{(t)} = \boldsymbol{x})dx_1 \ldots dx_{L_t} = \frac{1}{Z_c} \prod_{1 \le i < j \le L_t} (x_j^\eta - x_i^\eta)(x_j^\theta - x_i^\theta) \prod_{1 \le i \le L_t} w_c(x_i) dx_i$$

$$\mu_{L_t} = \frac{1}{L_t} \sum_{j=1}^{L_t} \delta_{x_j}$$

Natural think about Zelada's result, but no...

What is the relation between $\epsilon, L_t(N)$?

2 regimes

- If $\lim_{N\to\infty} N\epsilon(N) = 0$, then we recover the continuous case

- If $\lim_{N\to\infty} N\epsilon(N) = \beta > 0$, then $\mu(dx) \leq \frac{1}{x\beta\kappa}$ [Das-Dimitrov, discrete beta ens.]

Constraints = problems

LDP result

$\mu_{L_t} = \frac{1}{L_t}\sum_{j=1}^{L_t} \delta_{x_j}$ satisfies a LDP in $\mathfrak{P}$ with speed N^2 and with a good rate function $J[\mu] = I[\mu] - \inf_{\mu \in \mathfrak{P}} I[\mu]$

Here $\mathfrak{P} = \left\{ \mu \in P((0,1)) \,\middle|\, \mu(dx) \leq \frac{1}{\beta \kappa x} \right\}$

LDP result

$$I[\mu] = \frac{1}{2}\iint \left(\ln|x^\theta - y^\theta| + \ln|x^\eta - y^\eta|\right) d\mu(x)d\mu(y) + \mathfrak{K}[\mu]$$

$$\mathfrak{P} = \left\{ \mu \in P((0,1)) \,\middle|\, \mu(dx) \leq \frac{1}{\beta\kappa x} \right\}$$

How to calculate $\mu(dx)$??

Different form

$x \to x^{\theta}$ or x^{η}

$$I[\mu] = \frac{1}{2}\iint \big(\ln|x^{\nu} - y^{\nu}| + \ln|x - y|\big)\, d\omega(x)d\omega(y) + \widetilde{\mathfrak{K}}[\omega]$$

$$\widetilde{\mathfrak{P}} = \left\{ \mu \in P((0,1)) \,\middle|\, \mu(dx) \le \frac{1}{\beta\kappa\theta x} \right\}$$

$\beta = 0, \nu > 0$ we follow Claeys, Romano - Charlier

See also Arno, Leslie, Peter, Don, Lun ...

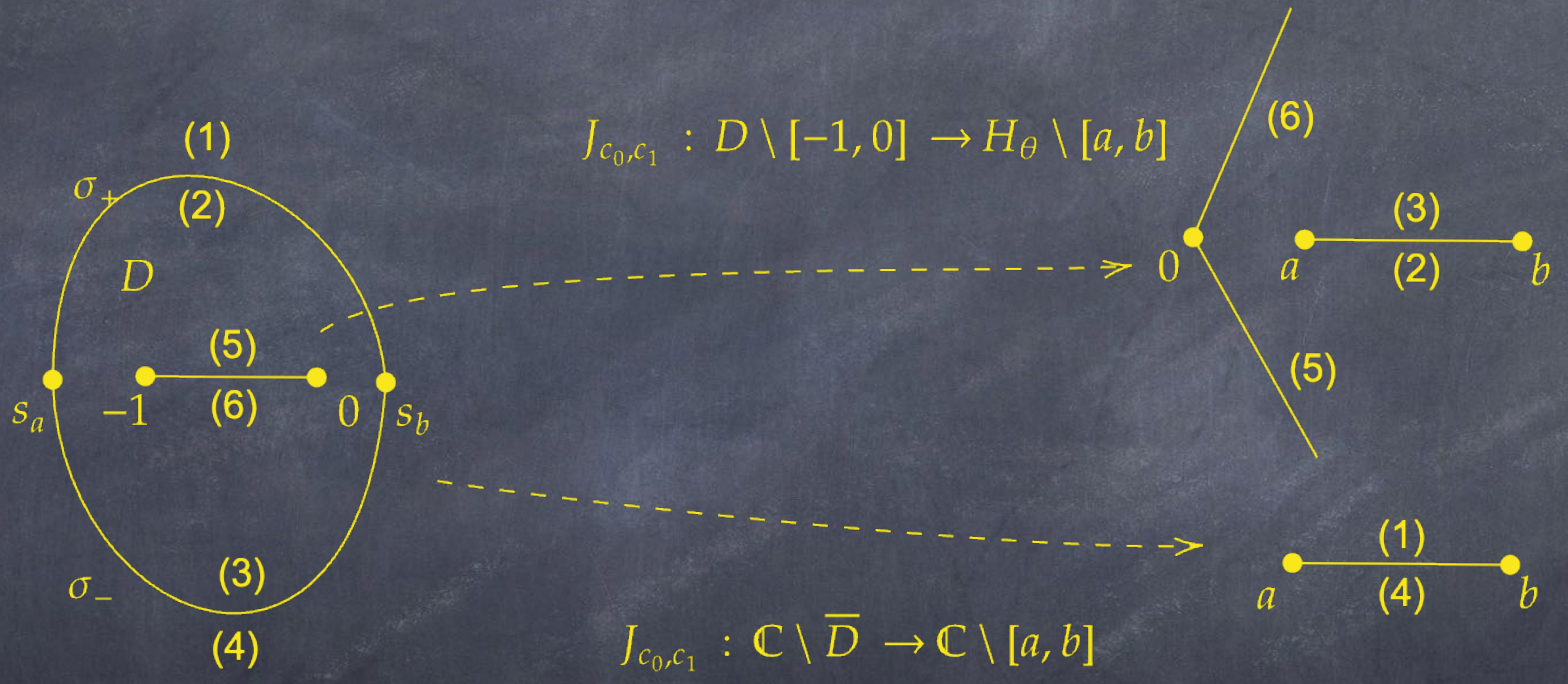
$$I[\omega] = \frac{1}{2}\iint \left(\ln|x^\nu - y^\nu| + \ln|x-y|\right) d\omega(x)d\omega(y) + \widetilde{\mathfrak{K}}[\omega]$$

$$g_1(z) = \int_{\text{supp}(\omega)} \ln(|z-y|)\omega(y)dy\,,\; g_\nu(z) = \int_{\text{supp}(\omega)} \ln(|z^\nu - y^\nu|\omega(y)dy$$

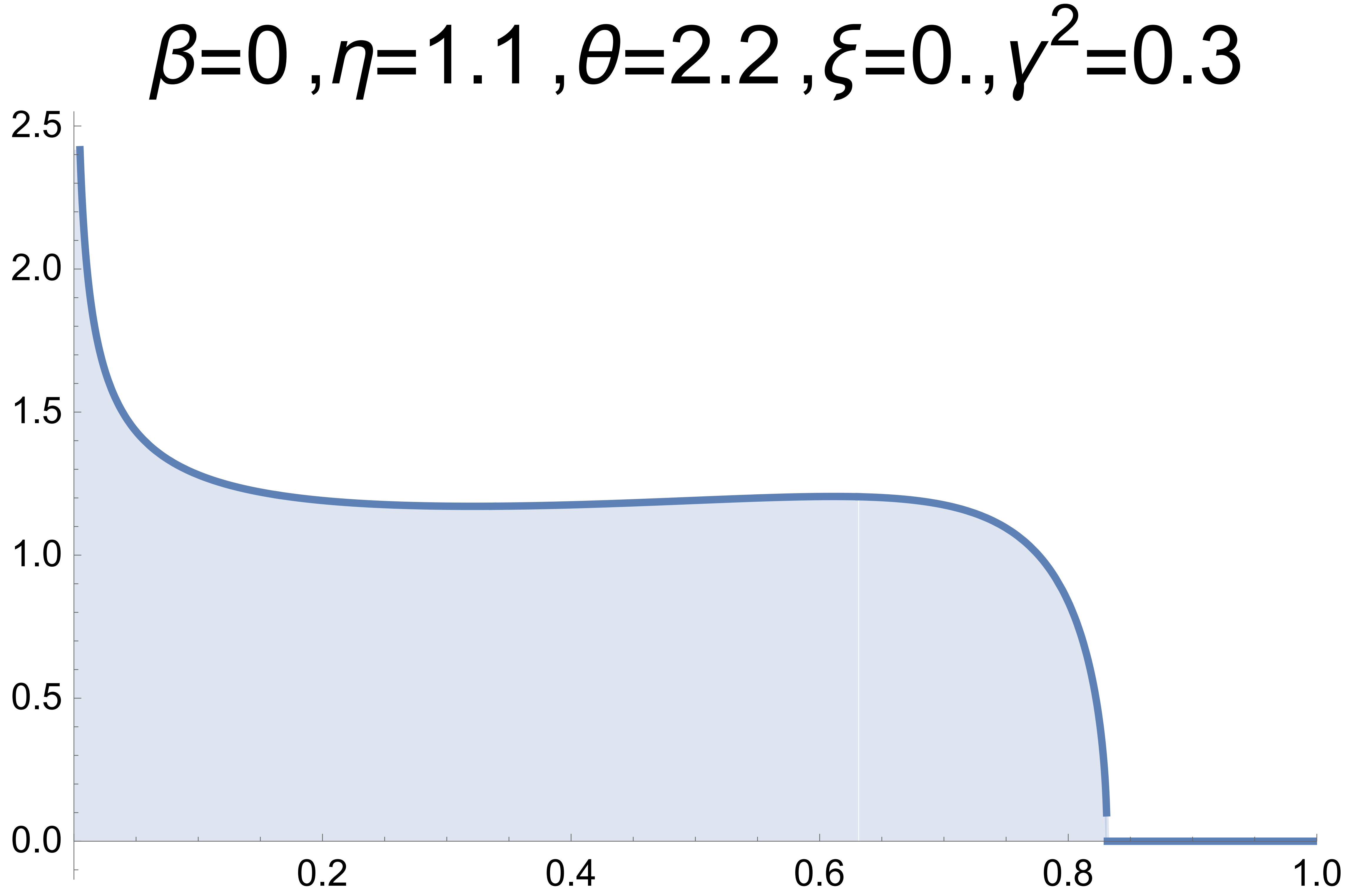
How??

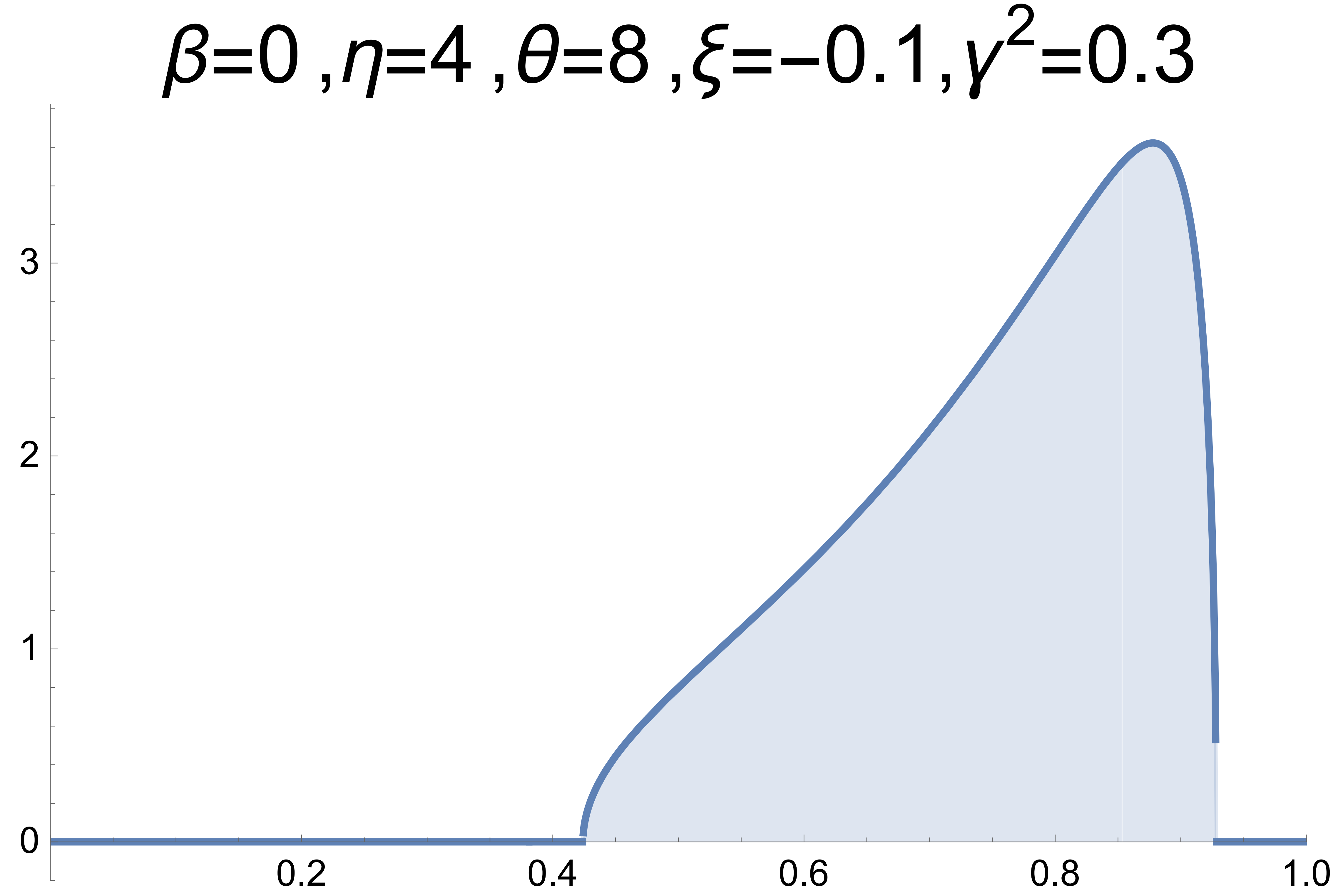
$$M(s)$$

$$J_{c_0,c_1}(s) = (c_1 s + c_0)\left(\frac{s+1}{s}\right)^{\frac{1}{\nu}}$$



$$g_1(z) = \int_{\mathrm{supp}(\omega)} \ln(|z-y|)\omega(y)dy\,,\; g_\nu(z) = \int_{\mathrm{supp}(\omega)} \ln(|z^\nu - y^\nu|\omega(y)dy$$


β=0 ,η=1.1 ,θ=2.2 ,ξ=0.,γ2=0.3
2.5
2.0
1.5
1.0
0.5
0.0
0.2
0.4
0.6
0.8
1.0


β=0 ,η=4 ,θ=8 ,ξ=−0.1,γ2=0.3
3
2
1
0
0.2
0.4
0.6
0.8
1.0

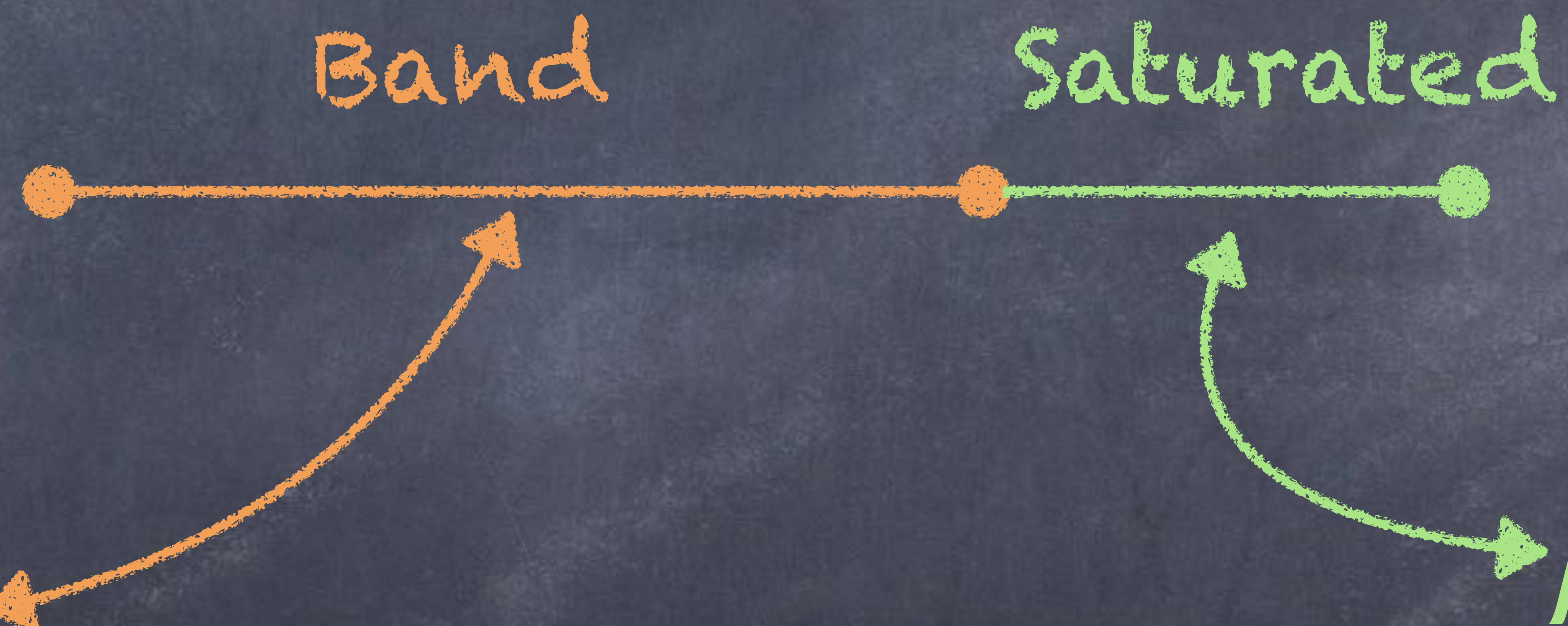
Decay at 0

$$\mu(x) \sim c_0\, x^{\frac{\theta\eta}{\theta+\eta}-1} \quad \text{as} \quad x \to 0$$

Wildly different from RMT, usually $\pm\frac{1}{2}$

$$\beta > 0$$

We cannot cover all the cases...

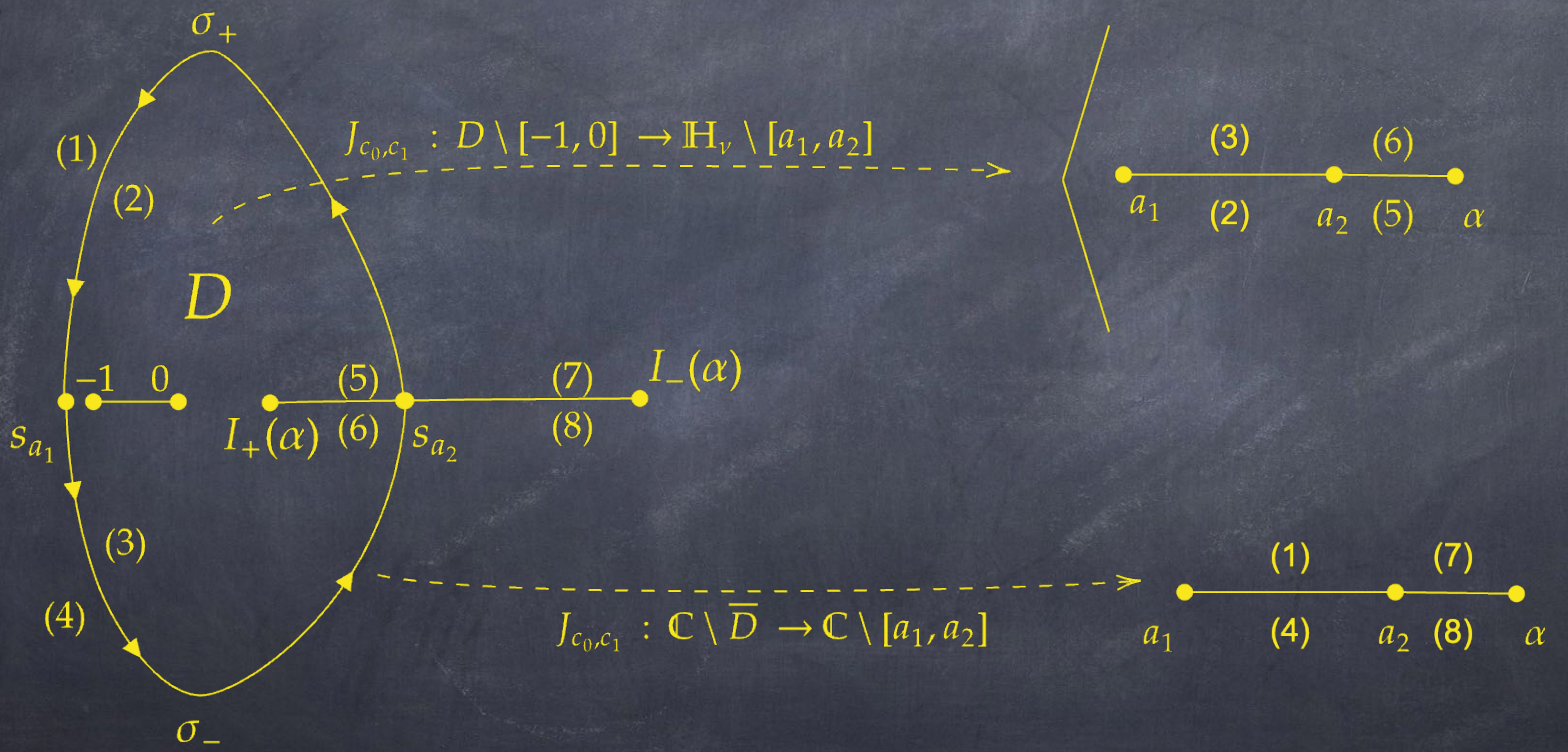


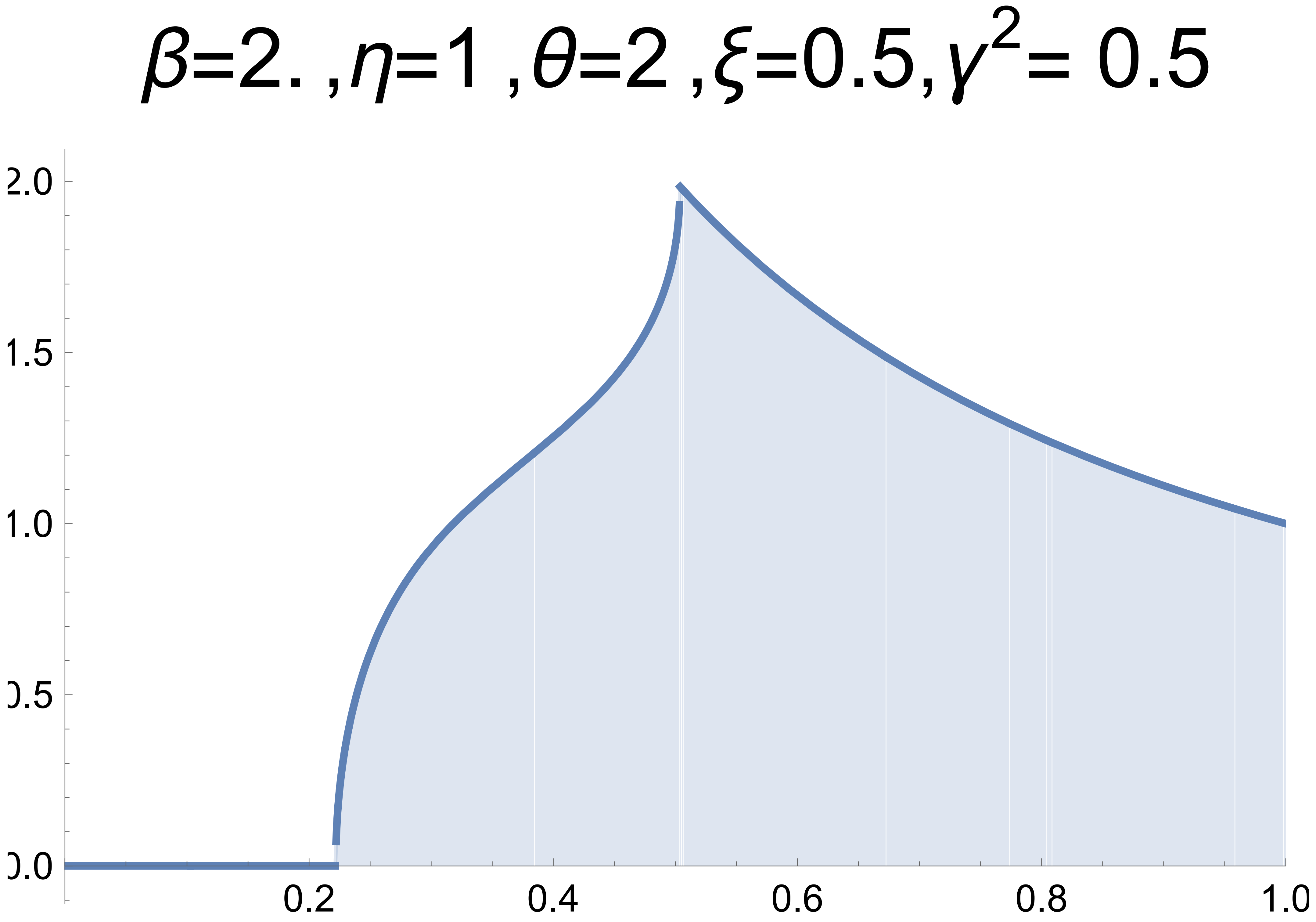
$$\mu(x) < \frac{1}{\beta \kappa x}$$

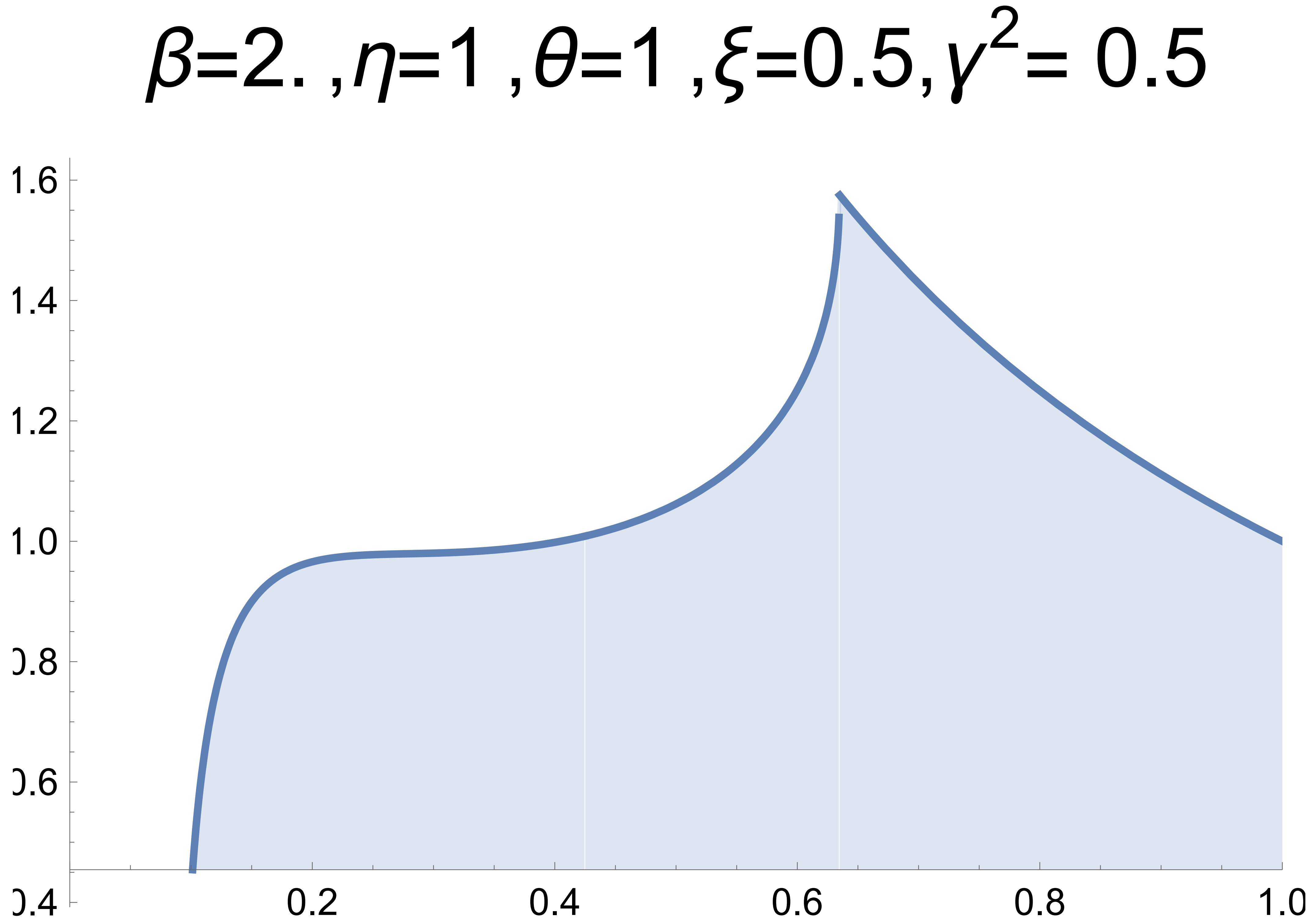
$$\mu(x) = \frac{1}{\beta \kappa x}$$

Discrete OP - the orange book

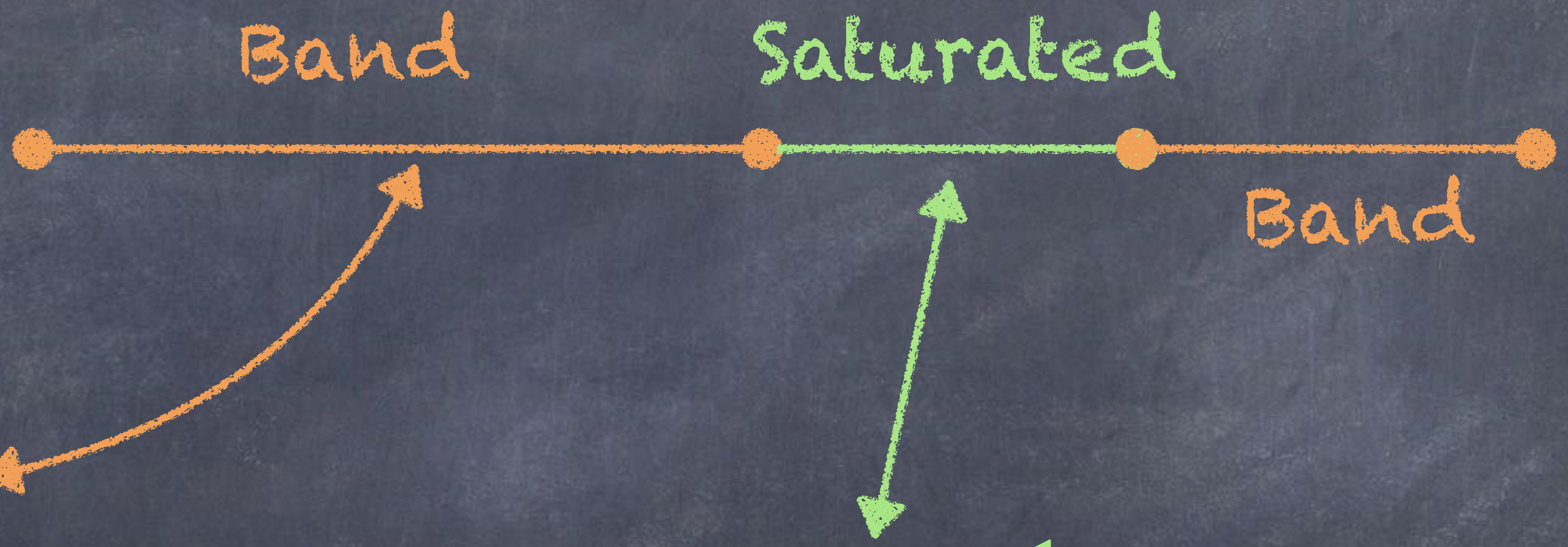
$$J_{c_0,c_1}(s) = (c_1 s + c_0)\left(\frac{s+1}{s}\right)^{\frac{1}{\nu}}$$




β=2. ,η=1 ,θ=2 ,ξ=0.5,γ2= 0.5
2.0
1.5
1.0
0.5
0.0
0.2
0.4
0.6
0.8
1.0


β=2. ,η=1 ,θ=1 ,ξ=0.5,γ2= 0.5
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.4
0.6
0.8
1.0

What's next?



$$\mu(x) < \frac{1}{\beta \kappa x}$$

$$\mu(x) = \frac{1}{\beta \kappa x}$$

Thank you for the attention
And happy B-day Peter