Entropic Cumulant Structures of Random State Ensembles

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Outline

Cumulant Structures of Hilbert-Schmidt Ensemble

Work in Progress: Cumulant Structures of Other Ensembles

Work in Perspectives: Algorithm to Analysis Gap

Cumulant Structures of Hilbert-Schmidt Ensemble*

*Huang-Wei [2025] Cumulant structures of entanglement entropy, available at arXiv:2502.05371

Cumulants of entropy

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Computing the first l cumulants of entanglement entropy

$$S = -\mathrm{tr}\left(\rho_A \ln \rho_A\right) = -\sum_{i=1}^{m} \lambda_i \ln \lambda_i$$

over the Hilbert-Schmidt ensemble

$$f(\boldsymbol{\lambda}) \propto \delta\left(1 - \sum_{i=1}^{m} \lambda_i\right) \prod_{1 \le i < j \le m} \left(\lambda_i - \lambda_j\right)^2 \prod_{i=1}^{m} \lambda_i^{n-m}$$

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can be converted to the first l cumulants of induced entropy

$$T = \sum_{i=1}^{m} x_i \ln x_i$$

over the Wishart-Laguerre ensemble

$$g(\boldsymbol{x}) \propto \prod_{1 \le i < j \le m} (x_i - x_j)^2 \prod_{i=1}^{m} x_i^{\alpha} \mathrm{e}^{-x_i}, \qquad \alpha = n - m$$

Moments conversion

Moments conversion

Lemma 1 The l-th moment of S can be recursively converted to the first l moments of T by

$$\mathbb{E}\left[S^l\right] = (-1)^l \frac{\Gamma(mn)}{\Gamma(mn+l)} \mathbb{E}\left[T^l\right] + \sum_{j=0}^{l-1} A_j \mathbb{E}\left[S^j\right],$$

where the coefficient A_j is

$$A_j = (-1)^{j+l+1} \binom{l}{j} B_{l-j}\left(\psi_0(mn+l), \ldots, \psi_{l-j-1}(mn+l)\right)$$

with $\psi_k(z)$ and $B_k(z_1, \ldots, z_k)$ respectively denoting the k-th polygamma functions

$$\psi_k(z) = \frac{\mathrm{d}^{k+1}}{\mathrm{d}z^{k+1}} \ln \Gamma(z) = \frac{\mathrm{d}^k}{\mathrm{d}z^k} \psi_0(z)$$

and the k-th complete exponential Bell polynomials.

Moments conversion

Moments conversion

Ideas of Lemma 1

- The change of variables

$$\lambda_i = \frac{x_i}{r}, \qquad r = \sum_{i=1}^{m} x_i$$

leads to the factorization of densities

$$g(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = h(r) f(\boldsymbol{\lambda})\,\mathrm{d}r\,\mathrm{d}\boldsymbol{\lambda}$$

and relations between linear statistics

$$S = r^{-1}\left(r \ln r - T\right), \qquad T = r\left(\ln r - S\right)$$

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$$S = r^{-1}\left(r \ln r - T\right), \qquad T = r\left(\ln r - S\right)$$

- Evaluating the integral over r in $\mathbb{E}\left[S^l\right]$ leads to Lemma 1

Cumulant structures: Overview

- The new methods uncover hidden cumulant structures that decouple each cumulant in a summation-free manner into its lower-order cumulants involving ancillary statistics

$$T_k = \sum_{i=1}^{m} x_i^k \ln x_i, \qquad R_k = \sum_{i=1}^{m} x_i^k$$

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- **Matrix-level results** enable the construction of a related but simpler cumulant that leads to a new decoupling structure through the Christoffel-Darboux kernel

$$K(x,y) \propto \sqrt{w(x)w(y)}\, \frac{L_{m-1}^{(\alpha)}(x)L_m^{(\alpha)}(y) - L_m^{(\alpha)}(x)L_{m-1}^{(\alpha)}(y)}{x-y}$$

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- **Kernel-level results** are more delicate tools to recycle the decoupled term produced from the new decoupling structure into lower-order cumulants

Cumulant structures: Initial data of mean $\kappa(T_k)$

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Lemma 2 The recurrence relation of mean formulas $\kappa(T_k)$ for $k \in \mathbb{R}_{\geq 0}$ is

$$\begin{aligned}(k+1)\kappa(T_k) &= (k-1)(2m+\alpha)\kappa(T_{k-1}) + m(m+\alpha) \times \\ &\quad \left(\kappa^{+}(T_{k-1}) - \kappa^{-}(T_{k-1})\right)\kappa(R_k) + (2m+\alpha)\kappa(R_{k-1}),\end{aligned}$$

where the initial value is

$$\kappa(T_0) = (m+\alpha)\psi_0(m+\alpha) - \alpha\psi_0(\alpha) - m$$

and

$$\kappa_l^{\pm}(\mathbf{X}) = \kappa_l(\mathbf{X})|_{m \to m \pm 1}.$$

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Example

$$\kappa(T_2) = m(m+\alpha)(2m+\alpha)\psi_0(m+\alpha) + \frac{m}{6}\left(10m^2 + 9m\alpha + 6m + 3\alpha + 2\right)$$

Cumulant structures: Matrix-level results

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Proposition 1 For a set $\mathbf{X} = \{X_1, \ldots, X_l\}$ of l linear statistics

$$X_j = \sum_{i=1}^{m} f_j(x_i)$$

over the Wishart density, the joint cumulant $\kappa_l(\mathbf{X})$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\kappa_l(\mathbf{X}) = \kappa_{l+1}(\mathbf{X}, T_0)\,.$$

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Example

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} \kappa(T_2) = \kappa(T_2, T_0)$$

Cumulant structures: Matrix-level results

Remarks on Proposition 1

- The proof utilizes generating functions of $\kappa_I(\mathbf{X})$ and $\frac{\mathrm{d}}{\mathrm{d}\alpha}\kappa_I(\mathbf{X})$, and the fact that

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\det{}^{\alpha}\left(\mathbf{Z}\mathbf{Z}^{\dagger}\right) = T_0 \det{}^{\alpha}\left(\mathbf{Z}\mathbf{Z}^{\dagger}\right)$$

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- Proposition 1 permits the construction of the decoupling statistics such that the difference between the desired cumulant and the constructed one decouples the kernels through the Christoffel-Darboux kernel

Cumulant structures: Kernel-level results

- Integrals resulting from the new decoupling structure consist of three types $H_l(\mathbf{X})$, $h_l(\mathbf{X})$, and $D_l(\mathbf{X})$, which are integrals involving products of Laguerre polynomials $L_m^{(\alpha)}(x)L_m^{(\alpha)}(y)$, $L_{m-1}^{(\alpha)}(x)L_{m-1}^{(\alpha)}(y)$, and $L_m^{(\alpha)}(x)L_{m-1}^{(\alpha)}(y)$, respectively

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- One must be able to recast these remaining integrals into lower-order cumulants so that the process of relating cumulants of different orders could continue

Cumulant structures: Kernel-level results

Proposition 2 The integrals $H_l(\mathbf{X})$ and $h_l(\mathbf{X})$ are recast respectively to lower-order cumulants as

$$\begin{aligned} H_l(\mathbf{X}) &= \sum_{\{p_1,\ldots,p_i\}\in\mathcal{P}_L} \prod_{j=1}^{i} \left(\kappa^{+}_{|p_j|}\left(\mathbf{X}_{p_j}\right) - \kappa_{|p_j|}\left(\mathbf{X}_{p_j}\right)\right), \\ h_l(\mathbf{X}) &= -\sum_{\{p_1,\ldots,p_i\}\in\mathcal{P}_L} \prod_{j=1}^{i} \left(\kappa^{-}_{|p_j|}\left(\mathbf{X}_{p_j}\right) - \kappa_{|p_j|}\left(\mathbf{X}_{p_j}\right)\right). \end{aligned}$$

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$$h_l(\mathbf{X}) = - \sum_{\{p_1,\dots,p_i\}\in\mathcal{P}_L} \prod_{j=1}^{i} \left(\kappa^{-}_{|p_j|}\left(\mathbf{X}_{p_j}\right) - \kappa_{|p_j|}\left(\mathbf{X}_{p_j}\right)\right).$$

Example

$$\begin{aligned} H(T_k) &= \frac{m!}{(m+\alpha)!}\int_0^\infty x^k \ln x w(x) L_m^{(\alpha)}(x) L_m^{(\alpha)}(x)\,\mathrm{d}x \\ &= \kappa^{+}(T_k) - \kappa(T_k) \end{aligned}$$

Cumulant structures: Kernel-level results

Recycling of $D_l(\mathbf{X})$ requires joint cumulant derivative

$$\kappa_l'(\mathbf{X}) = \kappa\big(X_1', \ldots, X_l\big) + \kappa\big(X_1, X_2', \ldots, X_l\big) + \cdots + \kappa\big(X_1, \ldots, X_l'\big),$$

$$X_j' = \sum_{i=1}^{m} x_i \frac{\mathrm{d}}{\mathrm{d}x_i} f_j(x_i)$$

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Proposition 3 The integral $D_l(\mathbf{X})$ is recast to lower-order cumulants as

$$D_l(\mathbf{X}) = \kappa_l'(\mathbf{X}).$$

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$$D_l(\mathbf{X}) = \kappa_l'(\mathbf{X}).$$

Example

$$\begin{aligned} D_1(T_k) &= -\frac{m!}{(m-1+\alpha)!} \int_0^\infty x^k \ln x w(x) L_{m-1}^{(\alpha)}(x) L_m^{(\alpha)}(x)\,\mathrm{d}x \\ &= k\kappa(T_k) + \kappa(R_k) \end{aligned}$$

Cumulant structures: Main results

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Theorem 1 For any $l \geq 2$, the joint cumulant $\kappa_l(T_k, T, \ldots, T)$ admits the decoupling structure

$$\kappa_l(T_k, T, \ldots, T) - \frac{\mathrm{d}}{\mathrm{d}\alpha}\kappa_{l-1}(T_{k+1}, T, \ldots, T) = \delta_l(k),$$

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where the decoupled term

$$\delta_l(k) = \sum_{s=1}^{l-1} \frac{(l-2)!}{(s-1)!(l-s-1)!} \left(\kappa(R)H_{l,s}(k) - D_{l,s}(k)\right)$$

consists of lower-order cumulants $H_{l,s}(k)$ and $D_{l,s}(k)$.

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Remarks on Theorem 1

- The proof of Theorem 1 is based on a proper combination of matrix-level and kernel-level results through the combinatorial structure of joint cumulants

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Cumulant structures: Main results

Remarks on Theorem 1

- The proof of Theorem 1 is based on a proper combination of matrix-level and kernel-level results through the combinatorial structure of joint cumulants
- Theorem 1 guarantees the existence of a closed-form cumulant formula $\kappa_l(T)$ for any order l, which implies the presence of anomalies is not necessary
- The existence also provides an explicit construction in generating the closed-form expression of $\kappa_l(T)$ for a given l

Cumulant structures: Implementation

Algorithm: Calculating l-th Cumulant $\kappa_l(T)$

Input: Any positive integer $l \geq 2$
$\kappa(T_l)$ closed-form expression

Output: Closed-form formula of $\kappa_l(T)$

$L \leftarrow 2$
while $L \leq l$ **do**
  $k \leftarrow l - L + 1$
  $\delta_L(k) \leftarrow$ by Theorem 1
  $\kappa_L(T_k, T, \ldots, T) \leftarrow \delta_L(k) + \frac{\mathrm{d}}{\mathrm{d}\alpha}\kappa_{L-1}(T_{k+1}, T, \ldots, T)$
  $L \leftarrow L + 1$
end while

Cumulant structures: A consequence of Theorem 1

Corollary 1 In the l-th cumulant $\kappa_l(S)$, terms involving polygamma function of highest order ψ_{l-1} are

$$(-1)^{l-1}\left(\psi_{l-1}(mn) - \frac{\kappa(R_l)}{(mn)_l}\psi_{l-1}(n)\right).$$

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$$(-1)^{l-1}\left(\psi_{l-1}(mn)-\frac{\kappa(R_l)}{(mn)_l}\psi_{l-1}(n)\right).$$

Remark Despite Theorem 1 generates $\kappa_l(S)$ expression for a given l, it is unable to provide highest-order polygamma terms for any l as captured in this corollary

Work in Progress: Cumulant Structures of Other Ensembles

Entanglement estimation

- Estimating the degree of entanglement of **bipartite model***

*Page [1993] Average entropy of a subsystem, *Phys. Rev. Lett.*

Entanglement estimation

- Estimating the degree of entanglement of **bipartite model***
  - measured by different **entanglement metrics**
    - von Neumann entropy (entanglement entropy)
    - quantum purity
    - Rényi entropy

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Entanglement estimation

- Estimating the degree of entanglement of **bipartite model***
  - measured by different **entanglement metrics**
    - von Neumann entropy (entanglement entropy)
    - quantum purity
    - Rényi entropy
  - over different models of **generic states**
    - Hilbert-Schmidt ensemble (Laguerre ensemble)
    - Bures-Hall ensemble (Cauchy-Laguerre ensemble)
    - fermionic Gaussian ensemble (Jacobi ensemble)

*Page [1993] Average entropy of a subsystem, *Phys. Rev. Lett.*

Bures-Hall and fermionic-Gaussian ensembles

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- Bures-Hall ensemble*†

$$f(\boldsymbol{\lambda}) \propto \delta\left(1-\sum_{i=1}^{m}\lambda_i\right)\prod_{1\leq i<j\leq m}\frac{(\lambda_i-\lambda_j)^2}{\lambda_i+\lambda_j}\prod_{i=1}^{m}\lambda_i^{n-m-\frac{1}{2}}$$

*Bertola et al [2014] Cauchy-Laguerre two-matrix model and the Meijer-G random point field, *Commun. Math. Phys.*

†Forrester-Kieburg [2016] Relating the Bures measure to the Cauchy two-matrix model, *Commun. Math. Phys.*

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- fermionic-Gaussian ensemble‡

$$f(\boldsymbol{\lambda}) \propto \prod_{1\le i<j\le m}\left(\lambda_i^{\gamma}-\lambda_j^{\gamma}\right)^2\prod_{i=1}^{m}(1-\lambda_i)^a(1+\lambda_i)^b, \quad \gamma=1,2$$

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‡Bianchi-Hackl-Kieburg [2021] The Page curve for fermionic Gaussian states, *Phys. Rev. B*

Results by existing methods: Bures-Hall ensemble

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- **Mean**: conjectured by Sarkar-Kumar'19*, proved in Wei'20†

$$\kappa_1 = \psi_0\left(mn - \frac{m^2}{2} + 1\right) - \psi_0\left(n + \frac{1}{2}\right)$$

*Sarkar-Kumar [2019] Bures-Hall ensemble: spectral densities and average entropies, *J. Phys. A*

†Wei [2020] Proof of Sarkar-Kumar's conjectures on average entanglement entropies over the Bures-Hall ensemble, *J. Phys. A*

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$$\kappa_1 = \psi_0\left(mn - \frac{m^2}{2} + 1\right) - \psi_0\left(n + \frac{1}{2}\right)$$

- **Variance**‡:

$$\kappa_2 = -\psi_1\left(mn - \frac{m^2}{2} + 1\right) + \frac{2n(2n+m) - m^2 + 1}{2n(2mn - m^2 + 2)}\psi_1\left(n + \frac{1}{2}\right)$$

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- **Skewness**§:

$$\kappa_3 = \psi_2\left(mn - \frac{m^2}{2} + 1\right) + c_1\psi_2\left(n + \frac{1}{2}\right) + c_2\psi_1\left(n + \frac{1}{2}\right)$$

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§Wei-Huang-Wei [2025] Skewness of von Neumann entropy over Bures-Hall random states, available at arXiv:2506.06663

Results by existing methods: fermionic-Gaussian ensemble

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- **Mean**: Bianchi-Hackl-Kieburg'21* and Bianchi-Hackl-Kieburg-Rigol-Vidmar'22†

$$\kappa_1 = a_1\psi_0(2m+2n) + a_2\psi_0(m+n) + a_2\psi_0(2n) + a_3\psi_0(n) + a_4$$

*Bianchi-Hackl-Kieburg [2021] The Page curve for fermionic Gaussian states, *Phys. Rev. B*

†Bianchi-Hackl-Kieburg-Rigol-Vidmar [2022] Volume-law entanglement entropy of typical pure quantum states, *PRX Quantum*

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- **Variance**‡:

$$\begin{aligned}\kappa_2 \;=\; & b_1\psi_1(2m+2n) + b_2\psi_1(2n) + b_3\psi_1(m+n) + \\ & b_4\psi_1(n) + b_5\psi_0(2m+2n) + b_6\psi_0(2n)\end{aligned}$$

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‡Huang-Wei [2023] Entropy fluctuation formulas of fermionic Gaussian states, *Ann. Henri Poincaré*

Procedure of finding cumulant structures

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Procedure of finding cumulant structures

- Construct decoupling statistics starting from Christoffel-Darboux kernels
- Identify matrix-level consecutive cumulant relations through derivative w.r.t. parameters of matrix densities
- Recycle remaining integrals from the decoupling into lower-order cumulants

Work in Perspectives: Algorithm to Analysis Gap

Algorithm to Analysis Gap

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Theorem 1 provides an *algorithm* to straightforwardly generate the $\kappa_l(T)$ expression for any given l. However, the mechanism that gives rise to each term in $\kappa_l(T)$ (except for the first term) is unknown *analytically* (even for the 'constant term'):

Algorithm to Analysis Gap

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$$\begin{aligned}
\kappa_1(T) &= a_1\psi_0(n) + m(m+1)/2 \\
\kappa_2(T) &= b_1\psi_1(n) + b_2\psi_0^2(n) + \cdots + m(m+1)/2 \\
\kappa_3(T) &= c_1\psi_2(n) + c_2\psi_0(n)\psi_1(n) + \cdots + m(m+1) \\
\kappa_4(T) &= d_1\psi_3(n) + d_2\psi_0(n)\psi_2(n) + \cdots + 3m(m+1)
\end{aligned}$$

Terms in blue are captured by Corollary 1; terms in red are conjectured to be $(l-1)!m(m+1)/2$; no clue about other terms

Happy Birthday, Peter!