Skewness of von Neumann Entropy over Bures-Hall Random States*

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*Joint work with Youyi Huang and Lu Wei, available at arXiv:2506.06663

Outline

Introduction and Main Result

Entanglement estimation

- Estimating the degree of entanglement of **bipartite model**[†]

[†] Page [1993] Average entropy of a subsystem, *Phys. Rev. Lett.*

Entanglement estimation

- Estimating the degree of entanglement of **bipartite model**†
  - measured by different **entanglement metrics**
    - $\boxed{\text{von Neumann entropy (entanglement entropy)}}$
    - quantum purity
    - Rényi entropy

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  - measured by different **entanglement metrics**
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    - quantum purity
    - Rényi entropy
  - over different models of **generic states**
    - Hilbert-Schmidt ensemble (Laguerre ensemble)
    - Bures-Hall ensemble (Cauchy-Laguerre ensemble)
    - fermionic Gaussian ensemble (Jacobi ensemble)

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The von Neumann entropy of a bipartite system of Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ with dimension m and n, respectively, is defined as

$$S = -\mathrm{tr}\left(\rho_A \ln \rho_A\right) = -\sum_{i=1}^{m} \lambda_i \ln \lambda_i$$

where ρ_A is computed by partial tracing of the full density matrix over the other subsystem B as $\rho_A = \mathrm{tr}_B(\rho)$.

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We compute the first three moments/cumulants of the entropy over the Bures-Hall ensemble

$$f\left(\boldsymbol{\lambda}\right) \propto \delta\left(1 - \sum_{i=1}^{m} \lambda_i\right) \prod_{1 \leq i < j \leq m} \frac{\left(\lambda_i - \lambda_j\right)^2}{\lambda_i + \lambda_j} \prod_{i=1}^{m} \lambda_i^{\alpha}$$

where

$$\alpha = n - m - \frac{1}{2}$$

Cumulant results of entropy

Denote the k-th order polygamma function as

$$\psi_k(z) = \frac{\mathrm{d}^{k+1}}{\mathrm{d}z^{k+1}} \ln \Gamma(z) = \frac{\mathrm{d}^k}{\mathrm{d}z^k} \psi_0(z)$$

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- **Mean**: conjectured by Sarkar-Kumar'19*, proved in Wei'20†

$$\kappa_1 = \psi_0\left(mn - \frac{m^2}{2} + 1\right) - \psi_0\left(n + \frac{1}{2}\right)$$

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- **Variance**‡:

$$\kappa_2 = -\psi_1\left(mn - \frac{m^2}{2} + 1\right) + \frac{2n(2n+m) - m^2 + 1}{2n(2mn - m^2 + 2)}\psi_1\left(n + \frac{1}{2}\right)$$

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Main result

- **Third Cumulant (relative to Skewness)**[§]:

$$\kappa_3 = \psi_2\left(mn - \frac{m^2}{2} + 1\right) + c_1\psi_2\left(n + \frac{1}{2}\right) + c_2\psi_1\left(n + \frac{1}{2}\right)$$

where

$$\begin{aligned} c_1 &= \frac{4m^2 - 8mn - 4n^2 - 7}{(2mn - m^2 + 2)(2mn - m^2 + 4)} \\ c_2 &= \frac{2(m^2 - 1)\left((m - 2n)^2 - 1\right)\left(-2m^2 + 4mn - 12n^2 + 7\right)}{n\left(2mn - m^2 + 2\right)^2\left(2mn - m^2 + 4\right)\left(4n^2 - 1\right)} \end{aligned}$$

[§] Wei-Huang-Wei [2025] Skewness of von Neumann entropy over Bures-Hall random states, available at arXiv:2506.06663

CLT conjecture

It is conjectured* in the limit

$$m \to \infty, \quad n \to \infty, \quad \frac{m}{n} = c \in (0, 1],$$

the standardized von Neumann entropy

$$X = \frac{S - \kappa_1}{\sqrt{\kappa_2}}$$

converges in distribution to Gaussian random variable $\mathcal{N}(0, 1)$

$$\psi_j(x) = \Theta\left(\frac{1}{x^j}\right), \quad x \to \infty, \quad j \geq 1,$$

Take $m = cn$ in the results above

$$\kappa_2 = \Theta\left(\frac{1}{n^2}\right), \quad \kappa_3 = \Theta\left(\frac{1}{n^4}\right)$$

$$\kappa_3^{(X)} = \frac{\kappa_3}{\kappa_2^{3/2}} = \frac{\Theta\left(\frac{1}{n^4}\right)}{\Theta\left(\frac{1}{n^3}\right)} = \Theta\left(\frac{1}{n}\right)$$

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Calculation of Third Cumulant

Moment conversion

To calculate the cumulants/moments over the Bures-Hall ensemble, a standard way is to calculate these over the unconstrained Bures-Hall ensemble

$$h(\boldsymbol{x}) \propto \prod_{1 \leq i < j \leq m} \frac{(x_i - x_j)^2}{x_i + x_j} \prod_{i=1}^{m} x_i^{\alpha} \mathrm{e}^{-x_i}$$

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which are related by factorization

$$h(\boldsymbol{x}) \prod_{i=1}^{m} \mathrm{d}x_i = f(\boldsymbol{\lambda}) g(\theta)\, \mathrm{d}\theta \prod_{i=1}^{m} \mathrm{d}\lambda_i$$

where

$$g(\theta) = \frac{1}{\Gamma\left(d\right)} \mathrm{e}^{-\theta} \theta^{d-1}$$

is the density of trace of the unconstrained ensemble

$$\theta = \sum_{i=1}^{m} x_i \qquad \theta \in [0, \infty)$$

Moment conversion

Consider von Neumann entropy of unconstrained Bures-Hall ensemble

$$T = \sum_{i=1}^{m} x_i \ln x_i$$

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$$T = \sum_{i=1}^{m} x_i \ln x_i$$

By change of variable

$$\lambda_i = \frac{x_i}{\theta}, \qquad \theta = \sum_{i=1}^{m} x_i$$

$$S^3 = -\theta^{-3} T^3 + 3S^2 \ln\theta - 3S \ln^2\theta + \ln^3\theta$$

We have moment conversion

$$\begin{aligned}\mathbb{E}_f\left[S^3\right] \quad = \quad & -\frac{1}{(d)_3}\mathbb{E}_h\left[T^3\right] + 3\psi_0(d+3)\mathbb{E}_f\left[S^2\right] \\ & -3\left(\psi_1(d+3)+\psi_0^2(d+3)\right)\mathbb{E}_f[S] \\ & +\left(\psi_2(d+3)+3\psi_1(d+3)\psi_0(d+3)+\psi_0^3(d+3)\right)\end{aligned}$$

Correlation functions

The k-point ($k \leq m$) correlation function of such an ensemble follows a Pfaffian point process[†], and the corresponding correlation kernels are related to those of Cauchy-Laguerre biorthogonal ensemble[‡]:

[†] Forrester-Kieburg [2016] Relating the Bures measure to the Cauchy two-matrix model, *Commun. Math. Phys.*

[‡] Bertola [2014] Cauchy–Laguerre two-matrix model and the Meijer-G random point field, *Commun. Math. Phys.*

Correlation functions

The k-point ($k \leq m$) correlation function of such an ensemble follows a Pfaffian point process†, and the corresponding correlation kernels are related to those of Cauchy-Laguerre biorthogonal ensemble‡: $p_k(x)$, $q_k(y)$ are Cauchy-Laguerre biorthogonal polynomials,

$$K_{00}(x,y) = \sum_{k=0}^{m-1} \frac{1}{h_k} p_k(x) q_k(y)$$

$$K_{01}(x,y) = -x^{\alpha} e^{-x} \sum_{k=0}^{m-1} \frac{1}{h_k} p_k(y) Q_k(-x)$$

$$K_{10}(x,y) = -y^{\alpha+1} e^{-y} \sum_{k=0}^{m-1} \frac{1}{h_k} P_k(-y) q_k(x)$$

$$K_{11}(x,y) = x^{\alpha} y^{\alpha+1} e^{-x-y} \sum_{k=0}^{m-1} \frac{1}{h_k} P_k(-y) Q_k(-x) - \frac{x^{\alpha} y^{\alpha+1} e^{-x-y}}{x+y}$$

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Cauchy-Laguerre biorthogonal polynomials

$$\int_0^\infty \int_0^\infty p_k(x) q_l(y) \frac{x^\alpha y^{\alpha+1} e^{-x-y}}{x+y} \, \mathrm{d}x \, \mathrm{d}y = h_k \delta_{kl}$$

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$P_k(x)$, $Q_k(y)$ are Cauchy transform of $p_k(x)$, $q_k(y)$, respectively.

$$\begin{aligned}
p_k(x) =& (-1)^k \frac{\Gamma(k+1)\Gamma(2\alpha+k+2)\Gamma(\alpha+k+1)}{\Gamma(2\alpha+2k+2)} \\
& \times G^{1,1}_{2,3}\left(\begin{matrix} -2\alpha-k-1;\ k+1 \\ 0;\ -\alpha,\ -2\alpha-1 \end{matrix} \middle| x \right) \\
q_k(y) =& (-1)^k \frac{\Gamma(k+1)\Gamma(2\alpha+k+2)\Gamma(\alpha+k+2)}{\Gamma(2\alpha+2k+2)} \\
& \times G^{1,1}_{2,3}\left(\begin{matrix} -2\alpha-k-1;\ k+1 \\ 0; -\alpha-1,\ -2\alpha-1 \end{matrix} \middle| y \right) \\
P_k(x) =& (-1)^{k+1} \frac{2\alpha+2k}{\Gamma(k)\Gamma(\alpha+k)} G^{3,1}_{2,3}\left(\begin{matrix} -k;\ k+2\alpha \\ -1, \alpha-1, 2\alpha; \end{matrix} \middle| -x \right) \\
Q_k(y) =& (-1)^{k+1} \frac{2\alpha+2k}{\Gamma(k)\Gamma(\alpha+k+1)} G^{3,1}_{2,3}\left(\begin{matrix} -k;\ k+2\alpha \\ -1, \alpha, 2\alpha; \end{matrix} \middle| -y \right)
\end{aligned}$$

Cumulant integrals: kernel level results

Let $f(x) = x \ln x$, recall:

- **Mean**

$$\kappa_1^T = \frac{1}{2}\int_0^\infty f(x)\,(K_{01}(x,x) + K_{10}(x,x))\,\mathrm{d}x$$

- **Variance**

$$\kappa_2^T = \frac{1}{2}(\mathrm{I_A} - \mathrm{I_B} - \mathrm{I_C} + 2\mathrm{I_D})$$

where

$$\begin{aligned}
\mathrm{I_A} &= \int_0^\infty \int_0^\infty f^2(x)\,(K_{01}(x,x) + K_{10}(x,x))\,\mathrm{d}x \\
\mathrm{I_B} &= \int_0^\infty \int_0^\infty f(x)f(y)\; K_{01}(x,y)K_{01}(y,x)\,\mathrm{d}x\,\mathrm{d}y \\
\mathrm{I_C} &= \int_0^\infty \int_0^\infty f(x)f(y)\; K_{10}(x,y)K_{10}(y,x)\,\mathrm{d}x\,\mathrm{d}y \\
\mathrm{I_D} &= \int_0^\infty \int_0^\infty f(x)f(y)\; K_{00}(x,y)K_{11}(x,y)\,\mathrm{d}x\,\mathrm{d}y
\end{aligned}$$

Cumulant integrals: kernel level results

- **Third Cumulant**

$$\kappa_3^T = \frac{1}{2}\mathrm{I}_\mathrm{A} - \frac{3}{2}\mathrm{I}_\mathrm{B} + \frac{1}{8}\mathrm{I}_\mathrm{C} + \frac{1}{8}\mathrm{I}_\mathrm{D}$$

$$\begin{aligned}
\mathrm{I}_\mathrm{A} &= \int_0^\infty f^3(x)\left(K_{01}(x,x) + K_{10}(x,x)\right)\,\mathrm{d}x \\
\mathrm{I}_\mathrm{B} &= \mathrm{I}_\mathrm{B}^{(1)} + \mathrm{I}_\mathrm{B}^{(2)} + \mathrm{I}_\mathrm{B}^{(3)} + \mathrm{I}_\mathrm{B}^{(4)} \\
\mathrm{I}_\mathrm{C} &= 4\left(2\mathrm{I}_\mathrm{C}^{(1)} + \mathrm{I}_\mathrm{C}^{(2)} - 2\mathrm{I}_\mathrm{C}^{(3)} - \mathrm{I}_\mathrm{C}^{(4)} + 2\mathrm{I}_\mathrm{C}^{(5)} + \mathrm{I}_\mathrm{C}^{(6)} - 2\mathrm{I}_\mathrm{C}^{(7)} - \mathrm{I}_\mathrm{C}^{(8)}\right) \\
\mathrm{I}_\mathrm{D} &= 2\left(\mathrm{I}_\mathrm{D}^{(1)} + 3\mathrm{I}_\mathrm{D}^{(2)} + 3\mathrm{I}_\mathrm{D}^{(3)} + \mathrm{I}_\mathrm{D}^{(4)}\right)
\end{aligned}$$

E.g.

$$\begin{aligned}
\mathrm{I}_\mathrm{B}^{(1)} &= \int_0^\infty \int_0^\infty f^2(x) f(y) K_{01}(x,y) K_{01}(y,x)\mathrm{d}x\mathrm{d}y \\
\mathrm{I}_\mathrm{C}^{(1)} &= \int_0^\infty\int_0^\infty\int_0^\infty f(x)f(y)f(z)K_{00}(x,y)K_{01}(y,z)K_{11}(x,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
\mathrm{I}_\mathrm{D}^{(1)} &= \int_0^\infty\int_0^\infty\int_0^\infty f(x)f(y)f(z)K_{01}(x,y)K_{01}(y,z)K_{01}(z,x)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z
\end{aligned}$$

Cumulant integrals: computation method

To obtain the l-th cumulant κ_l^T, each integral is explicitly computed using the following three steps

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- **Decouple** Replacing every $K_{00}(x, y)$, $K_{11}(x, y)$ in the integrals with the (finite) summation form of Cauchy-Laguerre kernel given explicitly by the Cauchy-Laguerre biothorgonal polynomials. For $K_{01}(x, y)$ and $K_{10}(x, y)$, replace them by integral representation of such kernels (which again gives finite summation form).
- **Compute** Using up to l derivatives (w.r.t. β_i) of the integral of $x_i^{\beta_i}$ multiplying certain Cauchy-Laguerre kernels in variable x_i's.
- **Simplify** The bulk of calculation lies in the simplification of resulting i-nested sums in each integral, which is an increasingly tedious and case-by-case task for higher-order cumulants

Cumulant integrals: Example

For integrals $\mathrm{I}_{\mathrm{B}}^{(1)}$, $\mathrm{I}_{\mathrm{B}}^{(2)}$, and I_{D}, we also utilize

$$\begin{aligned} K_{01}(x,y) &= x^{2\alpha+1}\int_0^1 t^{2\alpha+1} H_\alpha(ty) G_{\alpha+1}(tx)\,\mathrm{d}t \\ K_{10}(x,y) &= y^{2\alpha+1}\int_0^1 t^{2\alpha+1} H_{\alpha+1}(tx) G_\alpha(ty)\,\mathrm{d}t \end{aligned}$$

where

$$\begin{aligned} H_q(x) &= G_{2,3}^{1,1}\left(\begin{matrix} -m-2\alpha-1; m \\ 0; -q, -2\alpha-1 \end{matrix} \middle| x \right) \\ G_q(x) &= G_{2,3}^{2,1}\left(\begin{matrix} -m-2\alpha-1; m \\ 0, -q; -2\alpha-1 \end{matrix} \middle| x \right) \end{aligned}$$

and known identity

$$\begin{aligned} &\int_0^1 x^{a-1} G_{p,q}^{m,n}\left(\begin{matrix} a_1,\dots,a_n; a_{n+1},\dots,a_p \\ b_1,\dots,b_m; b_{m+1},\dots,b_q \end{matrix} \middle| \eta x \right) \mathrm{d}x \\ = &\; G_{p+1,q+1}^{m,n+1}\left(\begin{matrix} 1-a, a_1,\dots,a_n; a_{n+1},\dots,a_p \\ b_1,\dots,b_m; b_{m+1},\dots,b_q, -a \end{matrix} \middle| \eta \right). \end{aligned}$$

Cumulant integrals: Example

$$
\begin{aligned}
&\int_0^\infty\int_0^\infty\int_0^\infty x^{\beta_1}y^{\beta_2}z^{\beta_3}\ K_{01}(x,y)K_{01}(y,z)K_{01}(z,x)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
=\ &\sum_{i=0}^{m-1}\sum_{j=0}^{m-1}\sum_{k=0}^{m-1}\frac{(-1)^{i+j+k}\prod_{l=i,j,k}\Gamma(l+m+2\alpha+2))}{\prod\limits_{l=i,j,k}(\Gamma(l+1)\Gamma(l+\alpha+1)\Gamma(l+2\alpha+2)\Gamma(m-l))} \\
&\times\int_0^\infty y^{i+2\alpha+\beta_2}G^{2,2}_{3,4}\left(\begin{matrix}-j-2\alpha-1,-m-2\alpha-1,m\\0,-\alpha-1,-2\alpha-1,-j-2\alpha-2\end{matrix}\ \middle|\, y\right)\mathrm{d}y \\
&\times\int_0^\infty x^{k+2\alpha+\beta_1}G^{2,2}_{3,4}\left(\begin{matrix}-i-2\alpha-1,-m-2\alpha-1,m\\0,-\alpha-1,-2\alpha-1,-i-2\alpha-2\end{matrix}\ \middle|\, x\right)\mathrm{d}x \\
&\times\int_0^\infty z^{j+2\alpha+\beta_3}G^{2,2}_{3,4}\left(\begin{matrix}-k-2\alpha-1,-m-2\alpha-1,m\\0,-\alpha-1,-2\alpha-1,-k-2\alpha-2\end{matrix}\ \middle|\, z\right)\mathrm{d}z
\end{aligned}
$$

Apply Mellin transform for Meijer-G function and

$$
\mathrm{I}_{\mathrm{D}}^{(1)}=\frac{\partial^3}{\partial\beta_1\partial\beta_2\partial\beta_3}\int_0^\infty\int_0^\infty\int_0^\infty x^{\beta_1}y^{\beta_2}z^{\beta_3}K_{01}(x,y)K_{01}(y,z)K_{01}(z,x)\mathrm{d}x\mathrm{d}y\mathrm{d}z\Big|_{\beta_1=1,\beta_2=1,\beta_3=1}
$$

Summation identities: Example

By taking partial derivatives of the summation forms of integrals above, one arrives at a number of summations involving existing and new summation anomalies.

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$$\sum_{k=1}^{m}\left(\frac{\psi_0(k+b)\psi_0(k+c)}{k+a}+\frac{\psi_0(k+a)\psi_0(k+c)}{k+b}+\frac{\psi_0(k+a)\psi_0(k+b)}{k+c}\right)$$

$$=\left(\frac{1}{b-a}+\psi_0(a)\right)\sum_{k=1}^{m}\frac{\psi_0(k+c)}{k+b}+\left(\frac{1}{c-a}+\psi_0(a)\right)\sum_{k=1}^{m}\frac{\psi_0(k+b)}{k+c}$$

$$+\left(\frac{1}{a-b}+\psi_0(b)\right)\sum_{k=1}^{m}\frac{\psi_0(k+c)}{k+a}+\left(\frac{1}{c-b}+\psi_0(b)\right)\sum_{k=1}^{m}\frac{\psi_0(k+a)}{k+c}$$

$$+\left(\frac{1}{a-c}+\psi_0(c+m)\right)\sum_{k=1}^{m}\frac{\psi_0(k+b)}{k+a}+\left(\frac{1}{b-c}+\psi_0(c+m)\right)\sum_{k=1}^{m}\frac{\psi_0(k+a)}{k+b}$$

$$+\ (\text{Closed Forms})$$

Summation identities: Example

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$$=\left(\frac{1}{b-a}+\psi_0(a)\right)\sum_{k=1}^{m}\frac{\psi_0(k+c)}{k+b}+\left(\frac{1}{c-a}+\psi_0(a)\right)\sum_{k=1}^{m}\frac{\psi_0(k+b)}{k+c}$$
$$+\left(\frac{1}{a-b}+\psi_0(b)\right)\sum_{k=1}^{m}\frac{\psi_0(k+c)}{k+a}+\left(\frac{1}{c-b}+\psi_0(b)\right)\sum_{k=1}^{m}\frac{\psi_0(k+a)}{k+c}$$
$$+\left(\frac{1}{a-c}+\psi_0(c+m)\right)\sum_{k=1}^{m}\frac{\psi_0(k+b)}{k+a}+\left(\frac{1}{b-c}+\psi_0(c+m)\right)\sum_{k=1}^{m}\frac{\psi_0(k+a)}{k+b}$$
$$+\ (\text{Closed Forms})$$

$+$ 14 more new summation identities.

Closed-form result

One sums up into closed form

$$\begin{aligned}\kappa_3^T \quad = \quad & m(2n-m)\left(b_1\psi_2\left(n+\frac{1}{2}\right)+b_2\psi_0\left(n+\frac{1}{2}\right)\psi_1\left(n+\frac{1}{2}\right)\right.\\ & \left.+b_3\psi_1\left(n+\frac{1}{2}\right)+\psi_0^3\left(n+\frac{1}{2}\right)+\frac{9}{2}\psi_0^2\left(n+\frac{1}{2}\right)+3\psi_0\left(n+\frac{1}{2}\right)\right)\end{aligned}$$

where

$$b_1=\frac{-4m^2+8mn+4n^2+7}{8}$$

$$b_2=\frac{3(-m^2+2mn+4n^2+1)}{4n}$$

$$b_3=\frac{-m^4-16m^2n^2+4m^2n+5m^2+24mn^3-10mn+24n^4+10n^2-4}{2n(2n-1)(2n+1)}$$

New Challenges in Simplification, Re-summation Framework and Further Work

New Challenges in Simplification

Based on the summation method, the main effort to obtain the exact closed-form expression is to deal with the summation "anomalies" involving rational functions of polygamma functions that may not simplify to closed-form occuring in the calculation.

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Based on the summation method, the main effort to obtain the exact closed-form expression is to deal with the summation "anomalies" involving rational functions of polygamma functions that may not simplify to closed-form occuring in the calculation.

a takes the values m, $\alpha + m$, $2\alpha + m$, $2\alpha + 2m$

b, c takes the values 0, α, 2α, $2\alpha + m$

$$\Omega_1^{(a)} = \sum_{k=1}^{m} \frac{\psi_0(k)}{a+1-k} \qquad \Omega_2^{(a)} = \sum_{k=1}^{m} \frac{\psi_0(a+1-k)}{k}$$

$$\Omega_3^{(b,c)} = \sum_{k=1}^{m} \frac{\psi_0(k+b)}{(k+c)^2} \qquad \Omega_4^{(b,c)} = \sum_{k=1}^{m} \frac{\psi_0^2(k+b)}{k+c}$$

$$\Omega_5^{(b,c)} = \sum_{k=1}^{m} \frac{\psi_1(k+b)}{k+c} \qquad \Omega_6^{(b,c)} = \sum_{k=1}^{m} \frac{\psi_0(k+b)}{k+c}$$

New challenges in simplification

Table: New anomalies in the current work

$$\Omega_7^{(a)} = \sum_{k=1}^{m} \frac{\psi_0(k)}{(a+1-k)^2} \qquad \Omega_8^{(a)} = \sum_{k=1}^{m} \frac{\psi_0^2(k)}{a+1-k}$$

$$\Omega_9^{(a)} = \sum_{k=1}^{m} \frac{\psi_0(a+1-k)}{k^2} \qquad \Omega_{10}^{(a)} = \sum_{k=1}^{m} \frac{\psi_0^2(a+1-k)}{k}$$

$$\Omega_{11}^{(a)} = \sum_{k=1}^{m} \frac{\psi_0(k)\psi_0(k+a-m)}{m+1-k} \qquad \Omega_{12}^{(a)} = \sum_{k=1}^{m} \frac{\psi_0(k)\psi_0(a+1-k)}{m+1-k}$$

$$\Omega_{13}^{(a)} = \sum_{k=1}^{m} \frac{\psi_0(k)\psi_0(a+1-k)}{a+1-k} \qquad \Omega_{14}^{(a)} = \sum_{k=1}^{m} \frac{\psi_0(k)\psi_0(a+1-k)}{k}$$

$$\Omega_{15}^{(b)} = \sum_{k=1}^{m} \frac{\psi_0(k)\psi_0(k+b)}{k+b} \qquad \Omega_{16}^{(b)} = \sum_{k=1}^{m} \frac{\psi_0(k)\psi_0(k+b)}{k}$$

$$\Omega_{17}^{(a)} = \sum_{k=1}^{m} \frac{\psi_1(k)}{a+1-k} \qquad \Omega_{18}^{(a)} = \sum_{k=1}^{m} \frac{\psi_1(a+1-k)}{k}$$

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- Rewrite the argument with m appearing in one of its parameters, such that the summation anomaly $G(m)$ is a function with m as variable.
- Re-summation of $G(m)$ can be generated by iterating a suitably chosen recurrence relation

  $$G(m) = c_{m-1}G(m-1) + r_{m-1},$$

  Each iteration is to replace the term $G(m-i)$ with its previous one $G(m-i-1)$.
- Keep iterating until $G(m-i)$ vanishes, we then obtain an alternative form of $G(m)$

Re-summation framework: Example

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Sum over i on both side of

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use summation identities recursively breaks the anomaly $\Omega_{14}^{(a)}$ into $\Omega_4^{(a-m,0)}$, $\Omega_5^{(a-m,0)}$, $\Omega_4^{(0,a-m)}$, $\Omega_5^{(0,a-m)}$, $\Omega_6^{(a-m,0)}$ and closed forms.

Further work

- κ_4 relative to kurtosis of Bures-Hall ensemble
- Cumulant structure for Bures-Hall ensemble

Summation-free approach[§]

- Construct decoupling statistics starting from Christoffel-Darboux kernels

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Summation-free approach[§]

- Construct decoupling statistics starting from Christoffel-Darboux kernels
- Identify matrix-level consecutive cumulant relations through derivative w.r.t. parameters of matrix densities
- Recycle remaining integrals from the decoupling into lower-order cumulants

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Further work

$$\kappa_2^T = m(2n-m)\left(\psi_0\left(n+\frac{1}{2}\right)+\frac{1}{2}\psi_0^2\left(n+\frac{1}{2}\right)\right.$$
$$\left.+\frac{4n^2+2mn-m^2+1}{8n}\psi_1\left(n+\frac{1}{2}\right)\right)$$

where coefficient of $\psi_1\left(n+\frac{1}{2}\right)$ is the mean of induced purity (i.e. $\sum_{i=1}^{m} x_i^2$) over unconstrained Bures-Hall ensemble.

Further work

$$\begin{aligned}\kappa_2^T &= m(2n-m)\Bigg(\psi_0\left(n+\frac{1}{2}\right)+\frac{1}{2}\psi_0^2\left(n+\frac{1}{2}\right)\\ &\quad +\frac{4n^2+2mn-m^2+1}{8n}\psi_1\left(n+\frac{1}{2}\right)\Bigg)\end{aligned}$$

where coefficient of $\psi_1\left(n+\frac{1}{2}\right)$ is the mean of induced purity (i.e. $\sum_{i=1}^{m} x_i^2$) over unconstrained Bures-Hall ensemble.

$$\begin{aligned}\kappa_3^T &= m(2n-m)\Bigg(b_1\psi_2\left(n+\frac{1}{2}\right)+b_2\psi_0\left(n+\frac{1}{2}\right)\psi_1\left(n+\frac{1}{2}\right)\\ &\quad +b_3\psi_1\left(n+\frac{1}{2}\right)+\psi_0^3\left(n+\frac{1}{2}\right)+\frac{9}{2}\psi_0^2\left(n+\frac{1}{2}\right)+3\psi_0\left(n+\frac{1}{2}\right)\Bigg)\\ b_1 &= \frac{-4m^2+8mn+4n^2+7}{8}\end{aligned}$$

where coefficient of $\psi_2\left(n+\frac{1}{2}\right)$ is the mean of $\sum_{i=1}^{m} x_i^3$ over unconstrained Bures-Hall ensemble (numerical check).

Further work

Conjecture:

- $$\kappa_l^T = m(2n-m)\left(c_1\psi_{l-1}\left(n+\frac{1}{2}\right)+\cdots\right)$$

  where coefficient of $\psi_{l-1}\left(n+\frac{1}{2}\right)$ is the mean of $\sum_{i=1}^{m} x_i^l$ over unconstrained Bures-Hall ensemble.
- There is no non-polygamma term in κ_l^T.

Happy Birthday, Peter!