Multiple skew orthogonal polynomails and applications

Log-gases in Caeli Australi, Creswick, 4-15 August, 2025

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HAPPY BIRTHDAY
Peter

Facts

- Integrability in random matrix models has been a long-standing interests for studies in mathematical physics

1. Exact computations for correlation functions/Exact solutions for problems in statistical physics

combinatoric models with integrable kernels

— Forrester, Nagao & Rains 06' random involutions

— Nagao & Forrester 02' vicious random walkers

Facts

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  1. Exact computations for correlation functions/Exact solutions for problems in statistical physics

combinatoric models with integrable kernels

[Chapters 5, 6, 10 in the book "log-gases and random matrices"]

— Forrester, Nagao & Rains 06' random involutions

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Facts

- Integrability in random matrix models has been a long-standing interests for studies in mathematical physics

2. Asymptotic/Limiting behavior of correlation kernels and their applications in Painleve/Toda equations

[Chapters 7, 8, 9 in the book "Log-gases & random matrices"]

- Selberg integral (Chap. 4)
- symmetric function (Chap. 12)

……

Motivations

Skew orthogonal polynomials in random matrix theory

- The average characteristic polynomials of Orthogonal ensembles and Symplectic ensembles

- Characterization of some two-component log-gas system with different charges

Motivations

Skew orthogonal polynomials in random matrix theory

- The average characteristic polynomials of Orthogonal ensembles and Symplectic ensembles

– also in vicious random walkers, polynuclear growth model & Pfaffian Schur processes …

- Characterization of some two-component log-gas system with different charges

– with j.p.d.f. $\sim \prod_{j<k}^{N_1} |\alpha_j - \alpha_k| \prod_{j<k}^{N_2} |\beta_j - \beta_k|^4 \prod_{j=1}^{N_1} \prod_{k=1}^{N_2} |\alpha_j - \beta_k|^2 e^{-V_1(\alpha_j) - V_2(\beta_k)}$

$\alpha_j, \beta_k \in \mathbb{R} / S^1$ [See also Chap. 15 in the book "log-gases and random matrices"]

Motivations

Skew orthogonal polynomials and integrability

- Skew Christoffel-Darboux kernel

The correlation kernel of orthogonal / symplectic ensemble couldn't only be expressed as SOPs, but as a rank one perturbation of unitary ensemble.

[Adler, Forrester, Nagao & van Moerbeke, 99']

$$\tilde{S}_4(x,y) = * \, S_2(x,y)\Big|_{N\to 2N} + \cdot \, P_{2N}(y)\int_x^{+\infty} * \, P_{2N-1}(t)\,dt$$

Motivations

Skew orthogonal polynomials and integrability

- Skew Christoffel-Darboux kernel

[Forrester & -L., 20'] discrete symplectic ensemble (discrete linear lattice & q-lattice)

[-L. Shen, Yu & Forrester, 25'] discrete orthogonal ensemble

Some other discrete ensembles could be found at:

On linear lattice: Borodin-Gorin-Guionnet, Gaussian asymptotics of discrete β-ensemble. Publ. Math. IHES 17'

On q-lattice: Olshanski, Macdonald-level extension of beta ensembles and large N transitions. CMP 22'

Motivations

Skew orthogonal polynomials and integrability

- Skew Christoffel-Darboux kernel

- Wave functions for Pfaff lattice

introduced time evolutions & found Pfaff lattice from SOPs

[Adler, Horozov&van Moerbeke, IMRN, 1999]

[Adler&van Moerbeke, Math. Ann., 2002] → Existence & uniqueness of Pfaffian τ-functions, Fay identity, and Virasoro constraints

[Adler&van Moerbeke, Duke Math J., 2002]

Connection between Toda & Pfaff lattices

Motivations

Multiple orthogonal polynomials in random matrix theory

- Hermitian matrix ensemble with external source

  [Brezin & Hikami, Zinn-Justin, Bleher & Kuijlaars, Desrosiers & Forrester, …]

- Non-intersecting Brownian motion with different starting point and ending point

  Aptekarev, Bleher & Kuijlaars, Daems & Kuijlaars $\longrightarrow$ MOPs of mixed type

Motivations

Multiple orthogonal polynomials in classical integrable systems

- Multi-component Toda system

Adler - van Moerbeke - Vanhaecke (CMP 09'):

use Cauchy transform to give a whole multi-component Toda hierarchy, and characterize the gap probability by using the differential equation w.r.t the end points of intervals ...

Motivations

Multiple orthogonal polynomials in classical integrable systems

- Multi-component Toda system

Álvarez-Fernández, Prieto & Mañas (Adv. Math. 10'):

Group decomposition of moment matrix to give dressing operators,

Lax pairs of multi-component Toda system.

References (about random matrices and integrable systems)

Random matrix models	Orthogonality	Integrable lattice hierarchy	References
Hermitian (unitarily invariant) ensemble	Standard orthogonality	1d-Toda lattice/KdV hierarchy	[Gerasimov et al 91'] [Adler&van Moerbeke 95']
Orthogonal/symplectic invariant ensemble	Skew OPs	Pfaff lattice/DKP hierarchy	[Adler, Horozov&van Moerbeke 99'] [Adler&van Moerbeke 02']
Multi-matrix Ensemble	String OPs	2d-Toda hierarchy	[Adler&van Moerbeke 01']
Cauchy two-matrix model	Cauchy bi-orthogonal polynomials	CKP/C-Toda hierarchy	[SHL&Li 19']
Bures ensemble	Partial skew orthogonal polynomials	BKP/B-Toda hierarchy	[Hu&SHL 17'] [Chang, He, Hu&SHL 18']

Questions

- Whether do we have a concept of “Multiple skew-orthogonal polynomials” from random matrix theory or non-intersecting Brownian motions?

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- Is there any connection between multiple skew-orthogonal polynomials and classical integrable systems?

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- Is there any connection between multiple skew-orthogonal polynomials and classical integrable systems?

Reference:

SHL, B. Shen, J. Xiang and G. Yu, Multiple Skew-Orthogonal Polynomials and 2-component Pfaff lattice hierarchy. Ann. Henri Poincare, 25 (2024) 3333-3370.

Skew-orthogonal polynomials

- Definition: Let $S(x,y)$ be a skew symmetric kernel, i.e. $S(x,y)=-S(y,x)$, then one can define a skew inner product $\langle f(x), g(y)\rangle = \int_{\mathbb{R}^2} f(x)\, S(x,y)\, g(y)\, w(x) w(y)\, dx\, dy$. A family of poly. $\{P_n(x)\}$ is called SOPs if

$$\langle P_{2n}(x), P_{2m}(y)\rangle = \langle P_{2n+1}(x), P_{2m+1}(y)\rangle = 0, \quad \langle P_{2n}(x), P_{2m+1}(y)\rangle = h_n \delta_{n,m}, \quad h_n > 0.$$

— OE: $S(x,y) = \mathrm{sgn}(y-x)$, SE: $S(x,y) = \delta'(y-x)$, Bures: $S(x,y) = \dfrac{y-x}{x+y}$

— valid for discrete measure as well

Skew-orthogonal polynomials

- Definition: $\langle P_{2n}(x), y^i \rangle = 0, \quad i = 0, 1, 2, \cdots, 2n$

$$\langle P_{2n+1}(x), y^i \rangle = 0, \quad i = 0, 1, 2, \cdots, 2n-1, 2n+1$$

$$P_{2n}(x) = x^{2n} + a_{2n,2n-1} x^{2n-1} + \cdots + a_{2n,0} \qquad P_{2n+1}(x) = x^{2n+1} + a_{2n+1,2n} x^{2n} + \cdots + a_{2n+1,0}$$

- Linear algebra technique and Pfaffian expressions:

$$\begin{pmatrix} m_{0,0} & \cdots & m_{0,2n} \\ m_{1,0} & \cdots & m_{1,2n} \\ \vdots & & \vdots \\ \underline{m_{2n-1,0}} & \cdots & m_{2n-1,2n} \end{pmatrix} \begin{pmatrix} a_{2n,0} \\ a_{2n,1} \\ \vdots \\ a_{2n,2n-1} \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \rightarrow P_{2n}(x) = \frac{1}{\tau_{2n}} \begin{vmatrix} m_{0,0} & \cdots & m_{0,2n-1} & m_{0,2n} \\ \vdots & & \vdots & \vdots \\ m_{2n-1,0} & \cdots & m_{2n-1,2n-1} & m_{2n-1,2n} \\ 1 & \cdots & x^{2n-1} & x^{2n} \end{vmatrix}$$

$\hookrightarrow m_{i,j} = \langle x^i, y^j \rangle$

Skew-orthogonal polynomials

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- Linear algebra technique and Pfaffian expressions:

$$P_{2n}(x) = \frac{1}{\tau_{2n}} \mathrm{pf} \begin{pmatrix} m_{0,0} & \cdots & m_{0,2n} & -1 \\ \vdots & & \vdots & \vdots \\ m_{2n,0} & \cdots & m_{2n,2n} & -x^{2n} \\ 1 & \cdots & x^{2n} & 0 \end{pmatrix} := \frac{1}{\tau_{2n}} \mathrm{pf}(0, \cdots, 2n, x)$$

$$\mathrm{pf}(i, j) = m_{ij}, \quad \mathrm{pf}(i, x) = x^i$$

Skew-orthogonal polynomials

- Definition: $\langle P_{2n}(x), y^i \rangle = 0, \quad i = 0, 1, 2, \cdots, 2n$

$$\langle P_{2n+1}(x), y^i \rangle = 0, \quad i = 0, 1, 2, \cdots, 2n-1, 2n+1$$

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- Linear algebra technique and Pfaffian expressions:

$$\tau_{2n} = \mathrm{pf}(0, 1, \cdots, 2n-1) = \frac{1}{(2n)!} \int_{\mathbb{R}^{2n}} \mathrm{pf}\left(S(x_i, x_j)\right)_{i,j=1}^{2n} \Delta_{2n}(x) \prod_{i=1}^{n} \omega(x_i)\, dx_i$$

(de Bruijn's formula, 1955)

Ref.: [Chang, He, Hu & SHL, (Partial)-skew-orthogonal polynomials and related integrable lattices with Pfaffian τ-functions, CMP, 18']

Multiple skew-orthogonal polynomials

- Definition:

Definition 2.3. Given a multi-index $\vec{v} = (v_1, \cdots, v_p)$ such that $|\vec{v}| = v_1 + \cdots + v_p$ is odd, if there are p different weights $(\omega_1, \cdots, \omega_p)$ supported on γ and $\mathbb{S}(x, y)$ is a skew symmetric function from $\gamma \times \gamma$ to $\mathbb{R}$ so that all moments are finite, then for a fixed integer $b \in \{1, 2, \ldots, p\}$, there exist multiple skew-orthogonal polynomials $R_1(x), \cdots, R_p(x)$ and $\tilde{R}_b(x)$, such that

$$\begin{gathered}\int_{\gamma\times\gamma}\left(\sum_{i=1}^{p} R_i(x)\omega_i(x)\right)\mathbb{S}(x,y)y^j\omega_k(y)dxdy = 0, \quad j = 0, \cdots, v_k - 1, \quad k = 1, \cdots, p, \\ \int_{\gamma\times\gamma}\left(\sum_{i=1}^{p} R_i(x)\omega_i(x)\right)\mathbb{S}(x,y)\left(\sum_{\substack{i=1\\ i\neq b}}^{p} R_i(y)\omega_i(y) + \tilde{R}_b(y)\omega_b(y)\right)dxdy = 1,\end{gathered} \tag{2.15}$$

where deg $R_i(x) \leq v_i - 1$ $(i = 1, \cdots, p)$, and deg $\tilde{R}_b(x) \leq v_b$. Here, we assume that coefficients in the highest-order terms of R_b and $\tilde{R}_b$ are the same.

Multiple skew-orthogonal polynomials

- Definition:

Definition 2.3. Given a multi-index $\vec{v} = (v_1, \cdots, v_p)$ such that $|\vec{v}| = v_1 + \cdots + v_p$ is odd, if there are p different weights $(\omega_1, \cdots, \omega_p)$ supported on γ and $\mathbb{S}(x, y)$ is a skew symmetric function from $\gamma \times \gamma$ to $\mathbb{R}$ so that all moments are finite, then for a fixed integer $b \in \{1, 2, \ldots, p\}$, there exist multiple skew-orthogonal polynomials $R_1(x), \cdots, R_p(x)$ and $\tilde{R}_b(x)$, such that

$R_{(v_1, \cdots, v_p)}$: linear combination of poly. × weight

$$\int_{\gamma \times \gamma} \underbrace{\left(\sum_{i=1}^{p} R_i(x) \omega_i(x) \right)}_{R_{(v_1,\cdots,v_p)}} \mathbb{S}(x, y) y^j \omega_k(y) dx dy = 0, \quad j = 0, \cdots, v_k - 1, \quad k = 1, \cdots, p,$$

$$\int_{\gamma \times \gamma} \left(\sum_{i=1}^{p} R_i(x) \omega_i(x) \right) \mathbb{S}(x, y) \underbrace{\left(\sum_{\substack{i=1 \\ i \neq b}}^{p} R_i(y) \omega_i(y) + \tilde{R}_b(y) \omega_b(y) \right)}_{\tilde{R}_{(v_1,\cdots,v_p)}} dx dy = 1, \tag{2.15}$$

$R^{(i)}_{(v_1, \cdots, v_p)}$: i-th component of polynomials

where $\deg R_i(x) \leq v_i - 1$ $(i = 1, \cdots, p)$, and $\deg \tilde{R}_b(x) \leq v_b$. Here, we assume that coefficients in the highest-order terms of R_b and $\tilde{R}_b$ are the same.

Multiple skew-orthogonal polynomials

- Pfaffian expressions:

① A determinantal expression for block matrices

② The use of determinant identity to Pfaffian expressions

e.g.: $R_{(n_1,n_2)}(x) = R^{(1)}_{(n_1,n_2)}(x)\omega_1(x) + R^{(2)}_{(n_1,n_2)}(x)\omega_2(x)$

$$= \mathrm{Pf}\begin{bmatrix} A^{(1,1)}_{n_1,n_1} & A^{(1,2)}_{n_1,n_2} & -\psi_1^T(x)\omega_1(x) \\ A^{(2,1)}_{n_2,n_1} & A^{(2,2)}_{n_2,n_2} & -\psi_2^T(x)\omega_2(x) \\ \psi_1(x)\omega_1(x) & \psi_2(x)\omega_2(x) & 0 \end{bmatrix}$$

Application 1: Non-intersecting Brownian motions

- Karlin-McGregor formula vs LGV (Lindstorm-Gessel-Viennot) lemma

Karlin-McGregor formula: Consider n independent copies with given starting points $a_1 < a_2 < \cdots < a_n$ at time $t=0$. Then the probability density of the event of finding the paths at points $x_1, \cdots, x_n$ at time $t>0$ without any two of them having intersected in the time interval $[0, t]$ is proportional to $\det\left[p(a_i, x_j; t)\right]_{i,j=1}^{n}$, where $p(x, y; t)$ is the transition probability density of the diffusion process.

Application 1: Non-intersecting Brownian motions

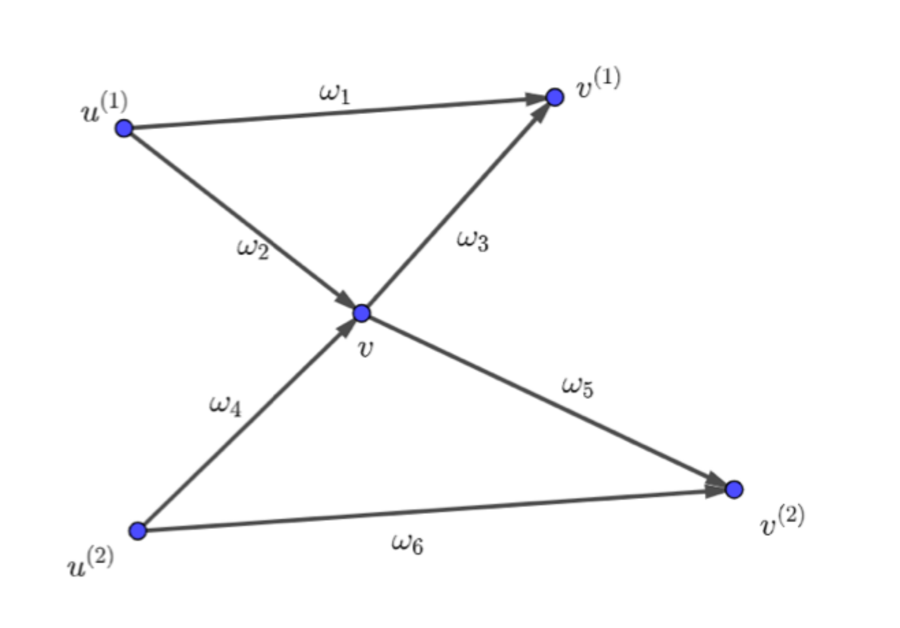
- Karlin-McGregor formula vs LGV (Lindstorm-Gessel-Viennot) lemma

LGV lemma: Let $\vec{u} = (u^{(1)}, \cdots, u^{(r)})$ and $\vec{v} = (v^{(1)}, \cdots, v^{(r)})$ be two ordered r-tuples of vertices in an acyclic directed graph D. If $\vec{u}$ and $\vec{v}$ are D-compitable, then

$$GF[\mathcal{P}_0(\vec{u};\vec{v})] = \det\left[h(u^{(i)}, v^{(j)})\right]_{i,j=1}^{r}.$$

Application 1: Non-intersecting Brownian motions

- Karlin-McGregor formula vs LGV (Lindstorm-Gessel-Viennot) lemma



$$GF\left[u^{(1)} \to v^{(1)},\ u^{(2)} \to v^{(2)}\right]$$

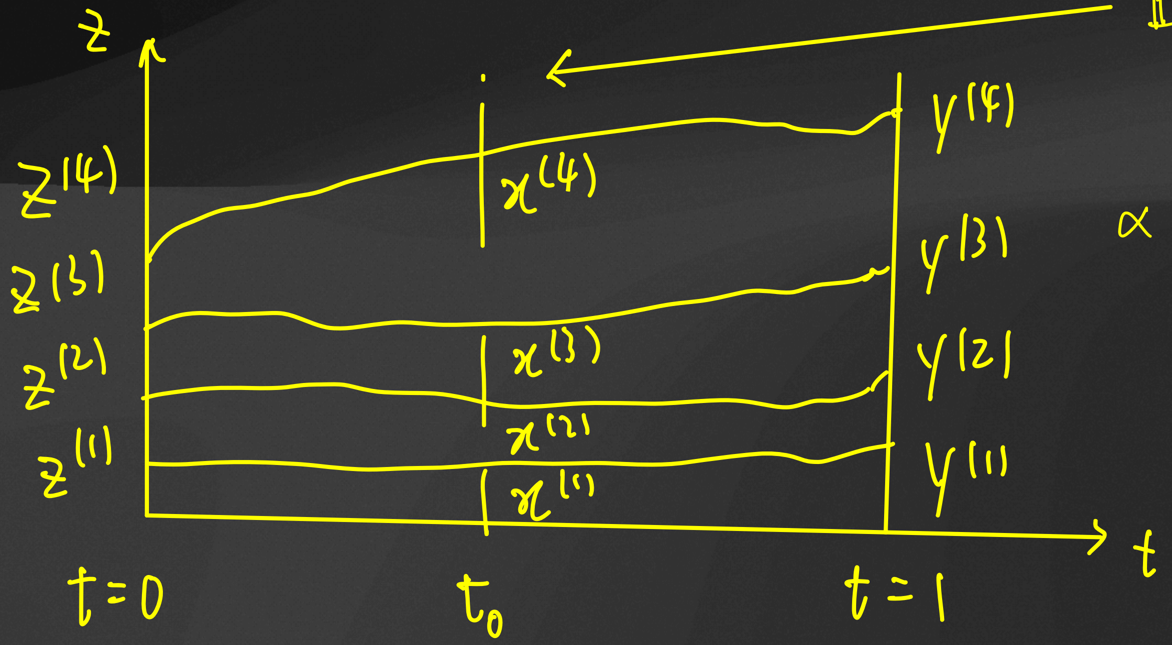
$$= \omega_1\omega_6 + \omega_1\omega_4\omega_5 + \omega_2\omega_3\omega_6$$

$$\det \begin{bmatrix} h(u^{(1)}, v^{(1)}) & h(u^{(1)}, v^{(2)}) \\ h(u^{(2)}, v^{(1)}) & h(u^{(2)}, v^{(2)}) \end{bmatrix}$$

$$= \det \begin{bmatrix} \omega_1 + \omega_2\omega_3 & \omega_2\omega_5 \\ \omega_3\omega_4 & \omega_4\omega_5 + \omega_6 \end{bmatrix}$$

Application 1: Non-intersecting Brownian motions

- Karlin-McGregor formula vs LGV (Lindstorm-Gessel-Viennot) lemma
- Applications into non-intersecting Brownian motions



$$\mathbb{P}\left[x^{(1)}, \dots, x^{(n)} \text{ at time } t_0 \;\middle|\; \begin{array}{l}\text{starts at } z^{(1)}, \dots, z^{(n)} \text{ at } t=0; \\ \text{ends at } y^{(1)}, \dots, y^{(n)} \text{ at } t=1\end{array} \right]$$

$$\propto \det\left[p(\vec{z}, \vec{x}, t_0)\right] \times \det\left[\underline{p(\vec{x}, \vec{y}, 1-t_0)}\right]$$

$$\hookrightarrow p(a,b,t) = \frac{1}{\sqrt{\pi t}} e^{-\frac{(b-a)^2}{2t}}$$

Application 1: Non-intersecting Brownian motions

- Karlin-McGregor formula vs LGV (Lindstorm-Gessel-Viennot) lemma
- Applications into non-intersecting Brownian motions
- Confluent cases and multiple orthogonal polynomials

$$\lim_{\substack{z^{(1)},\cdots,z^{(k)}\to z_1 \\ z^{(k+1)},\cdots,z^{(n)}\to z_2}} \det\left[p(\vec{z},\vec{x},t)\right] \sim \det\begin{bmatrix} x_i^{j-1} p(z_1,x_i,t) \\ x_i^{l-1} p(z_2,x_i;t) \end{bmatrix}_{\substack{j=1,\cdots,k \\ l=1,\cdots,n-k}}$$

$$\text{j.p.d.f.} \propto \Delta_n(x) \det\begin{bmatrix} x_i^{j-1} p(z_1,x_i,t) \\ x_i^{l-1} p(z_2,x_i,t) \end{bmatrix}$$ starting from z_1, z_2 and ending at 0

Multiple orthogonal polynomials of type I / II

Application 1: Non-intersecting Brownian motions

- Karlin-McGregor formula vs LGV (Lindstorm-Gessel-Viennot) lemma
- Applications into non-intersecting Brownian motions
- Confluent cases and multiple orthogonal polynomials

$$\lim_{\substack{z^{(1)},\cdots,z^{(k)}\to z_1 \\ z^{(k+1)},\cdots,z^{(n)}\to z_2}} \det\left[p(\vec{z},\vec{x},t)\right] \sim \det\begin{bmatrix} x_i^{j-1}\,p(z_1,x_i,t) \\ x_i^{l-1}\,p(z_2,x_i;t)\end{bmatrix} \quad \begin{matrix} j=1,\cdots,k \\ l=1,\cdots,n-k\end{matrix}$$

$$\text{j.p.d.f.} \propto \det\begin{bmatrix} x_i^{j_1-1}\,p(z_1,x_i,t) \\ \vdots \\ x_i^{j_l-1}\,p(z_l,x_i,t)\end{bmatrix} \times \det\begin{bmatrix} x_i^{q_1-1}\,p(x_i,y_1,1-t) \\ \vdots \\ x_i^{q_m-1}\,p(x_i,y_m,1-t)\end{bmatrix}$$

starting from $z_1,\cdots,z_l$
ending at $y_1,\cdots,y_m$

Multiple orthogonal polynomials of mixed type [Daems & Kuijlaars, 04']

Application 1: Non-intersecting Brownian motions

- Stembridge's formula

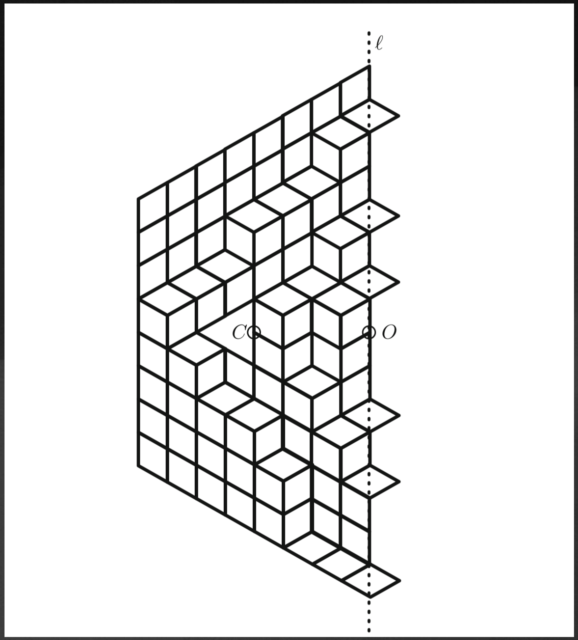
Let $\vec{u} = (u^{(1)}, \dots, u^{(2n)})$ be $2n$-tuple of vertices in an acyclic directed graph D. If $I \subset V$ is a totally ordered subset of vertices s.t $\vec{u}$ is D-compatible with I, then $GF[\mathcal{P}_0(\vec{u}; I)] = \mathrm{pf}\left[Q_I(u^{(i)}, u^{(j)})\right]_{i,j=1}^{2n}$,

where $Q_I(u^{(i)}, u^{(j)}) = \sum_{x, y \in I} h(u^{(i)}, x)\, h(u^{(j)}, y)\, \mathrm{sgn}(y - x)$.

Stembridge. Nonintersecting paths, Pfaffians, and plane partition. Adv. Math., 1992

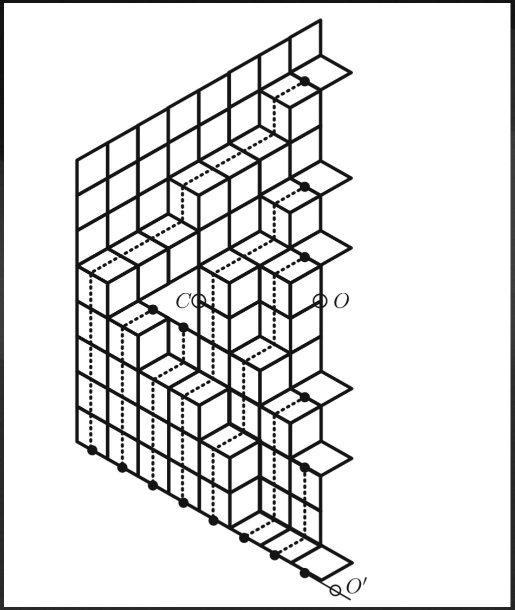
Application 1: Non-intersecting Brownian motions

- Stembridge's formula



$\vec{u} \to \vec{v} \oplus I$

$\Longrightarrow$



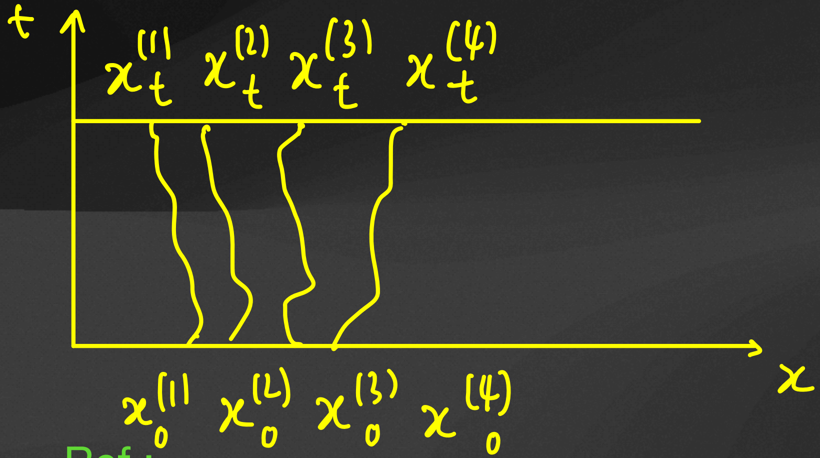
Ref.:

Ciucu and Krattenthaler, The interaction of a gap with a free boundary in a two dimensional dimer system, Commun. Math. Phys., 2011.

Application 1: Non-intersecting Brownian motions

- Stembridge's formula

- Applications into NIBM (distribution of ending points)



$$\mathbb{P}\left[x_t^{(1)}, \dots, x_t^{(n)} \;\middle|\; \text{starts at } x_0^{(1)}, \dots x_0^{(n)}\right]$$

$$\propto \det\left[p(\vec{x}_0, \vec{x}_t; t)\right]$$

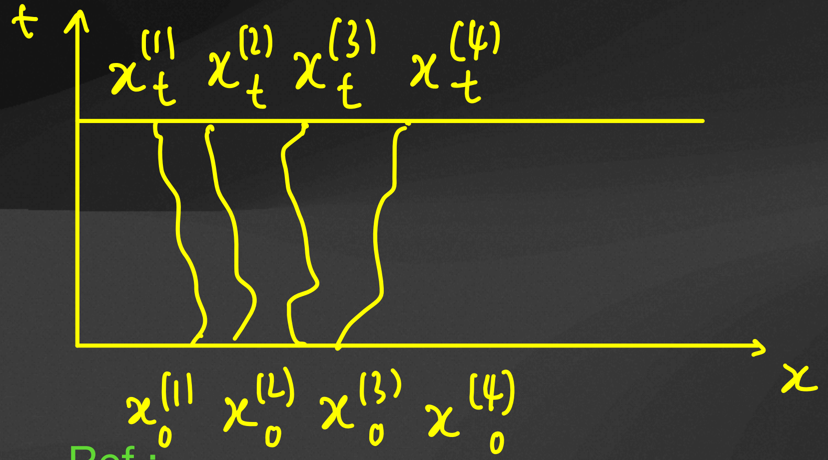
Ref.:

Katori and Tanemura, Scaling limits of vicious walks and two-matrix models, Phys. Rev. E, 2002.

Application 1: Non-intersecting Brownian motions

- Stembridge's formula

- Applications into NIBM (distribution of ending points)



$$\mathbb{P}\left[x_t^{(1)}, \cdots, x_t^{(n)} \,\middle|\, \begin{array}{c}\text{starts at} \\ x_0^{(1)}, \cdots x_0^{(n)}\end{array}\right]$$

$$\propto \det\left[p(\vec{x}_0, \vec{x}_t; t)\right]$$

Ref.:

Katori and Tanemura, Scaling limits of vicious walks and two-matrix models, Phys. Rev. E, 2002.

A discrete version was given by Forrester & Nagao, 2002.

Application 1: Non-intersecting Brownian motions

- A generalization of Stembridge's formula and multiple skew-orthogonal polynomials

Let $\vec{u} = (u^{(1)}, \cdots, u^{(r)})$ and $\vec{v} = (v^{(1)}, \cdots, v^{(s)})$ be sequences of vertices in an acyclic direct graph D. Assume that I & J are totally ordered subsets of V, s.t. $J \oplus \vec{u}$ and $\vec{v} \oplus I$ are D-compatible and separated.

If $r, s \in 2\mathbb{N}$, then $GF[\mathcal{P}_0(J \oplus \vec{u}; v \oplus I)] = \mathrm{pf}\begin{bmatrix} Q & H \\ -H^T & \tilde{Q} \end{bmatrix}$.

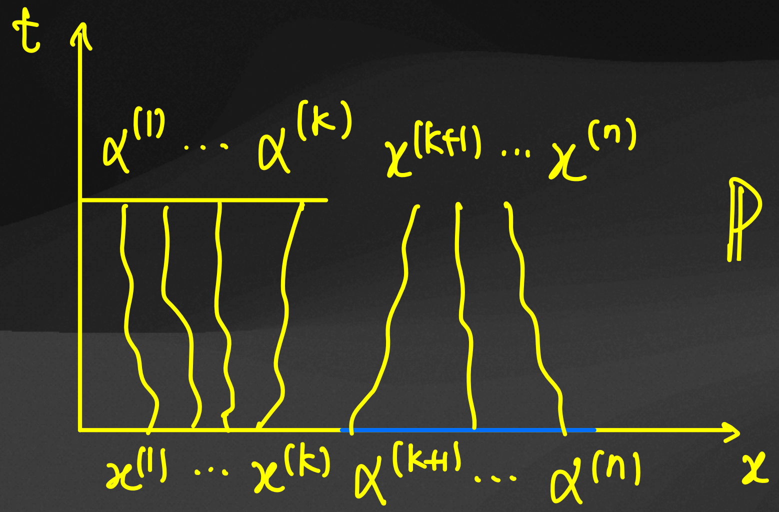
$$Q = [Q_I(u^{(i)}, u^{(j)})], \quad \tilde{Q} = [Q_J(v^{(i)}, v^{(j)})], \quad H = [h(u^{(i)}, v^{(s+1-j)})].$$

Ref.:

SHL and Z. Yao, Non-intersecting path explanation for block Pfaffians and applications into skew-orthogonal polynomials, Adv. Appl. Math., 2025.

Application 1: Non-intersecting Brownian motions

- A generalization of Stembridge's formula and multiple skew-orthogonal polynomials



$$\mathbb{P}\left(\begin{matrix} \alpha^{(1)}, \dots, \alpha^{(k)}, \\ \alpha^{(k+1)}, \dots, \alpha^{(n)} \end{matrix} \,\middle|\, \begin{matrix} \text{first } k \text{ points starts at } x^{(1)}, \dots, x^{(k)} \\ \text{the rest } n-k \text{ points ends at } x^{(k+1)}, \dots, x^{(n)} \end{matrix} \right)$$

$$\sim \det \left[p(\vec{\alpha}, \vec{x}; t) \right]$$

Ref.:
SHL and Z. Yao, Non-intersecting path explanation for block Pfaffians and applications into skew-orthogonal polynomials, Adv. Appl. Math., 2025.

Application 1: Non-intersecting Brownian motions

- A generalization of Stembridge's formula and multiple skew-orthogonal polynomials

$$\lim_{\substack{x^{(1)},\cdots,x^{(k)}\to x_1\\ x^{(k+1)},\cdots,x^{(n)}\to x_2}} \det\left[p(\vec{\alpha},\vec{x};t)\right] \quad \leftarrow \quad \mathbb{P}\left(\begin{matrix}\alpha^{(1)},\cdots,\alpha^{(k)},\\ \alpha^{(k+1)},\cdots,\alpha^{(n)}\end{matrix}\;\middle|\;\begin{matrix}\text{first } k \text{ points starts at } x^{(1)},\cdots,x^{(k)}\\ \text{the rest } n-k \text{ points ends at } x^{(k+1)},\cdots,x^{(n)}\end{matrix}\right)$$

$$\wr$$

$$\sim \det\left[p(\vec{\alpha},\vec{x};t)\right]$$

$$\det\begin{bmatrix}\alpha_j^{l-1}\,p(\alpha_j,x_1,t)\\ \alpha_j^{m-1}\,p(\alpha_j,x_2,t)\end{bmatrix}_{\substack{l=1,\cdots,k\\ m=1,\cdots,n-k}}$$

Ref.:

SHL and Z. Yao, Non-intersecting path explanation for block Pfaffians and applications into skew-orthogonal polynomials, Adv. Appl. Math., 2025.

Application 2: 2-component Pfaff lattice

- Time evolutions: Recall: $R_{(n_1,n_2)}(x) = R^{(1)}_{(n_1,n_2)}(x)\omega_1(x) + R^{(2)}_{(n_1,n_2)}(x)\omega_2(x)$

$$\omega_1(x;t) = \omega_1(x)\exp\left(\sum_{i=1}^{+\infty} t_i x^i\right) \qquad \omega_2(x;t) = \omega_2(x)\exp\left(\sum_{i=1}^{+\infty} s_i x^i\right)$$

$$\begin{cases} \partial_{t_1}\left(* R^{(1)}_{(n_1,n_2)}(x;t,s)\right) = *' \tilde{R}^{(1)}_{(n_1,n_2)}(x;t,s) \\ \partial_{s_1}\left(* R^{(2)}_{(n_1,n_2)}(x;t,s)\right) = *' \tilde{R}^{(2)}_{(n_1,n_2)}(x;t,s) \end{cases}$$

Ref.:

1. P. van Moerbeke. Nonintersecting Brownian motions, integrable systems and orthogonal polynomials, MSRI Publ., 2007.

2. M. Adler, P. van Moerbeke, and P. Vanhaecke. Moment Matrices and Multi-Component KP, with applications in random matrix theory, Comm. Math. Phys., 2009.

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Time evolutions are also imposed on the partition functions / normalizations / τ-functions

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Application 2: 2-component Pfaff lattice

- Time evolutions:

Evolutions on wave functions and τ- functions

- Discrete spectral transformations:

$$\tau_{(v_1,v_2-1)}d^{(2)}_{(v_1+1,v_2+1)}R^{(2)}_{(v_1+1,v_2+1)}(x)=\tau_{(v_1+1,v_2)}d^{(2)}_{(v_1,v_2)}\tilde{R}^{(2)}_{(v_1,v_2)}(x)$$
$$-\partial_{s_1}\tau_{(v_1+1,v_2)}d^{(2)}_{(v_1,v_2)}R^{(2)}_{(v_1,v_2)}(x)+\tau_{(v_1,v_2+1)}d^{(2)}_{(v_1+1,v_2-1)}R^{(2)}_{(v_1+1,v_2-1)}(x)$$

Relations between MSOPs with adjacent indexes.

—More examples about discrete spectral transformations of skew-orthogonal polynomials could be found at:

SHL, G. Yu, Christoffel transformations for (partial-)skew-orthogonal polynomials and applications, Adv. Math., 2024.

Application 2: 2-component Pfaff lattice

- Time evolutions:

Evolutions on wave functions and τ- functions $\Rightarrow$

Comparison of coefficients of polynomials

compatible condition of Lax pairs

$$D_{t_1}\tau_{(n_1,n_2-1)}\cdot\tau_{(n_1,n_2+1)}=D_{s_1}\tau_{(n_1+1,n_2)}\cdot\tau_{(n_1-1,n_2)}$$

$$D_{s_1}D_{t_1}\tau_{(n_1-1,n_2)}\cdot\tau_{(n_1-1,n_2)}=2\left(\tau_{(n_1,n_2-1)}\tau_{(n_1-2,n_2+1)}-\tau_{(n_1,n_2+1)}\tau_{(n_1-2,n_2-1)}\right)$$

$\cdots$

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Application 2: 2-component Pfaff lattice

- Cauchy/Stieltjes transform:

$$C_\psi g(z) = \int_{\gamma\times\gamma} \frac{\psi(z)}{x-z} S(x,y)\, g(y)\, dx\, dy \Rightarrow$$

$$\mathcal{C}_{\omega_1}\left(d^{(i)}_{(v_1,v_2)} R^{(i)}_{(v_1,v_2)}\right) = (-1)^{v_1} z^{-(v_1+1)} \tau_{(v_1+1,v_2)}(\mathbf{t}+[z^{-1}],\mathbf{s}),$$
$$\mathcal{C}_{\omega_2}\left(d^{(i)}_{(v_1,v_2)} R^{(i)}_{(v_1,v_2)}\right) = z^{-(v_2+1)} \tau_{(v_1,v_2+1)}(\mathbf{t},\mathbf{s}+[z^{-1}]).$$

- 2-component Pfaff lattice hierarchy:

$$(-1)^{u_1+v_1}\oint_{C_\infty} e^{\xi(t-t',z)} z^{v_1-u_1-2}\, \tau_{(v_1-1,v_2)}(t-[z^{-1}],s)\, \tau_{(u_1+1,u_2)}(t'+[z^{-1}],s')\, dz$$
$$+(-1)^{u_1+v_1}\oint_{C_\infty} e^{\xi(t-t',z)} z^{u_1-v_1-2}\, \tau_{(v_1+1,v_2)}(t+[z^{-1}],s)\, \tau_{(u_1-1,u_2)}(t'-[z^{-1}],s')\, dz$$
$$=\oint_{C_\infty} e^{\xi(s-s',z)} z^{v_2-u_2-2}\, \tau_{(v_1,v_2-1)}(t,s-[z^{-1}])\, \tau_{(u_1,u_2+1)}(t',s'+[z^{-1}])\, dz$$
$$+\oint_{C_\infty} e^{\xi(s-s',z)} z^{u_2-v_2-2}\, \tau_{(v_1,v_2+1)}(t,s+[z^{-1}])\, \tau_{(u_1,u_2-1)}(t',s'-[z^{-1}])\, dz$$

Remarks on Pfaff lattice hierarchy

- The term “Pfaff lattice hierarchy” was first proposed by Adler and van Moerbeke [M. Adler, E. Horozov and P. van Moerbeke, IMRN, 1999]. In fact, this equation is equivalent to the KP equation of D-type according to the classification by Kyoto school.

- The 2-component Pfaff lattice hierarchy was studied by Takasaki [K. Takasaki, Auxiliary linear problem, Difference Fay identities and dispersionless limit of the Pfaff-Toda lattice hierarchy, SIGMA, 2009]. The infinite-dimensional Lie algebraic realization of the general multi-component case is still unknown.

- An elliptic skew orthogonal polynomials and related Pfaff lattice (Landau-Lifshitz equation) was considered recently [W. Fu and SHL, Skew-orthogonal polynomials and Pfaff lattice hierarchy associated with an elliptic curve, IMRN, 2024].

Thanks for your attention!