Large deviations for the number of real eigenvalues of the elliptic GinOE

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Log-gases in Caeli Australi 2025
@ MATRIX Institute, Creswick

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joint work with Gernot Akemann & Sung-Soo Byun
arXiv:2503.18310

Outline

1. Literature review and main results
2. Sketch of proof

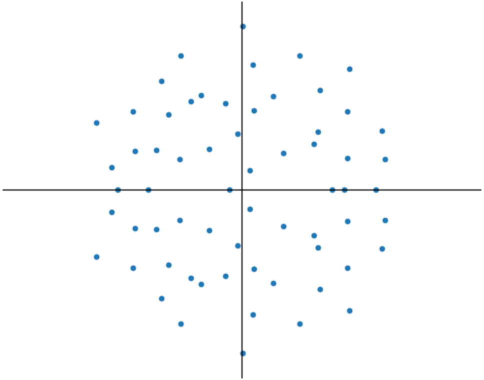
The Ginibre Orthogonal Ensemble (GinOE)

GinOE: An $n \times n$ matrix

$$G = (g_{jk})_{j,k=1}^{n},$$

where

$$g_{jk} \sim \mathrm{N}_{\mathbb{R}}(0, 1/n).$$



$n = 64$

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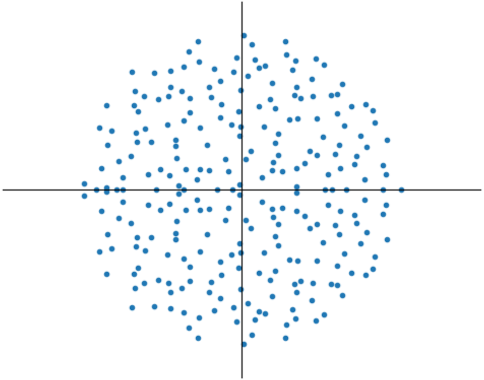

$n = 256$

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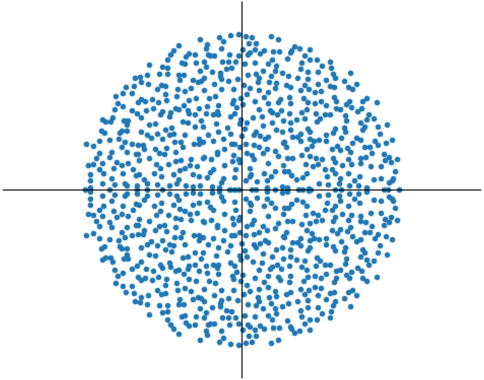
- The circular law.



$n = 1024$

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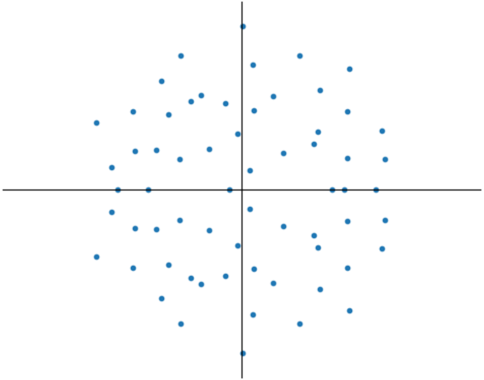

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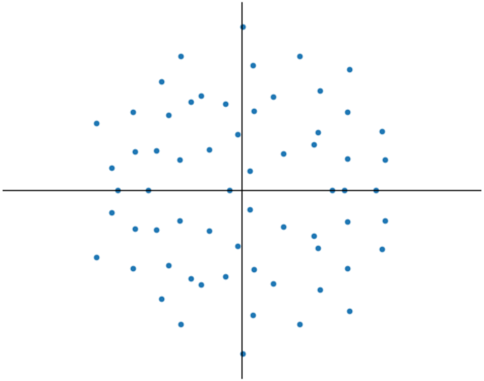


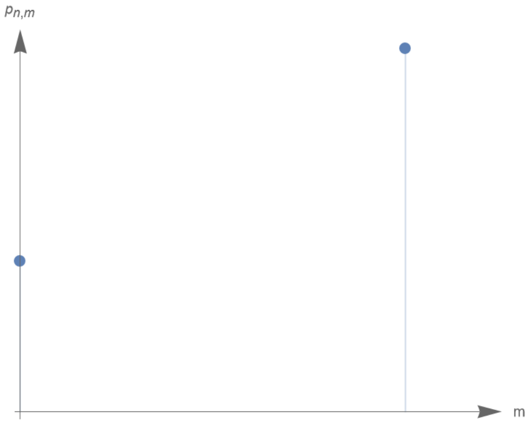
$n = 64$

The probability to have m real eigenvalues:

$$p_{n,m} := \mathbb{P}\Big[\mathcal{N}_n = m\Big],$$

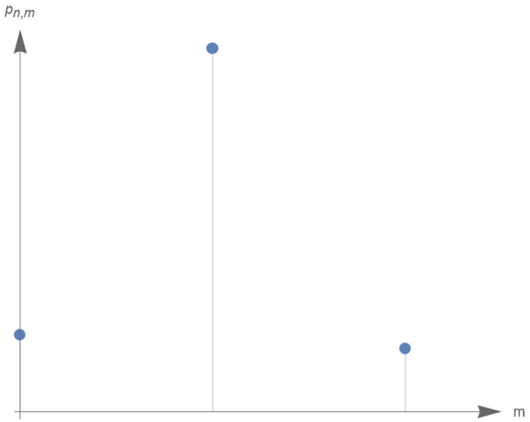
where $\mathcal{N}_n := \#\{\text{real eigenvalues of the GinOE}\}$.

The p.d.f of $\mathcal{N}_n$: Numerics



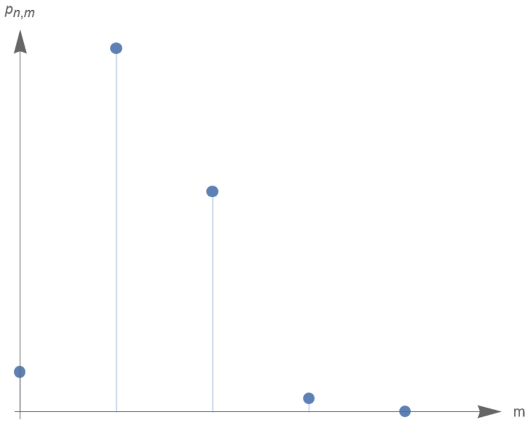
$n = 2$

The p.d.f of $\mathcal{N}_n$: Numerics



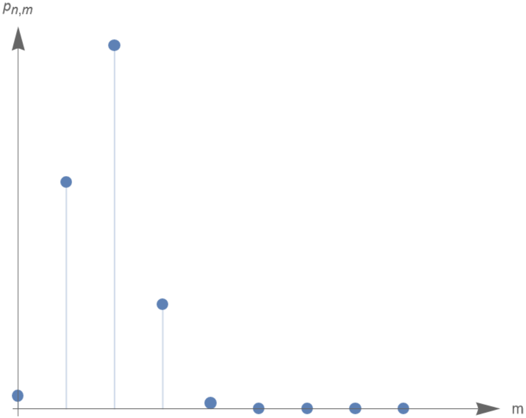
$n = 4$

The p.d.f of $\mathcal{N}_n$: Numerics



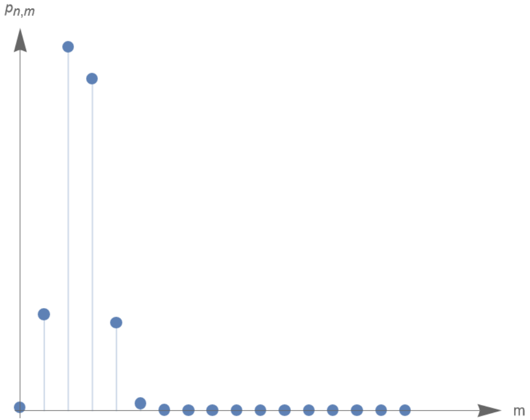
$n = 8$

The p.d.f of $\mathcal{N}_n$: Numerics



$n = 16$

The p.d.f of $\mathcal{N}_n$: Numerics



$n = 32$

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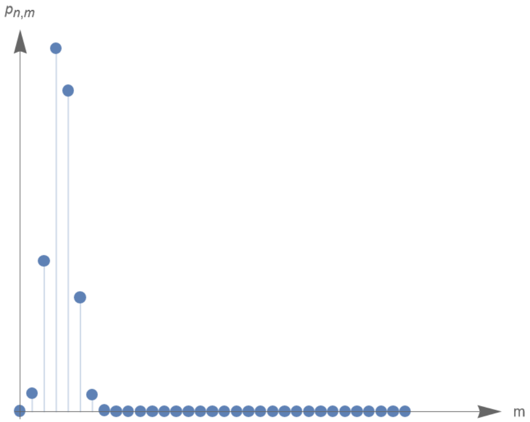

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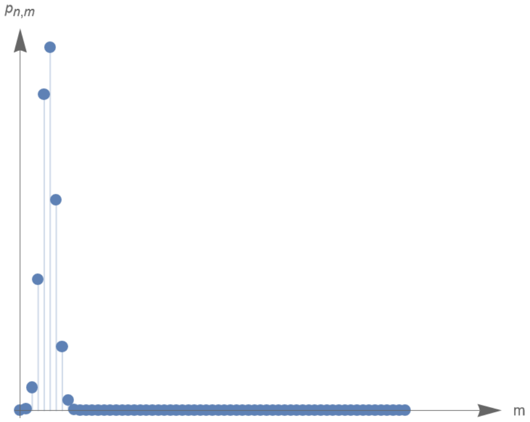

$n = 128$

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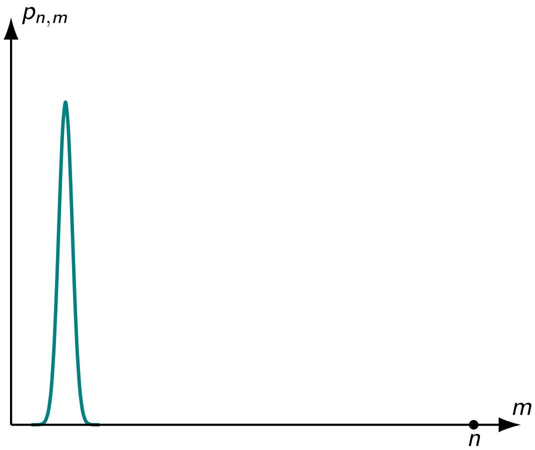
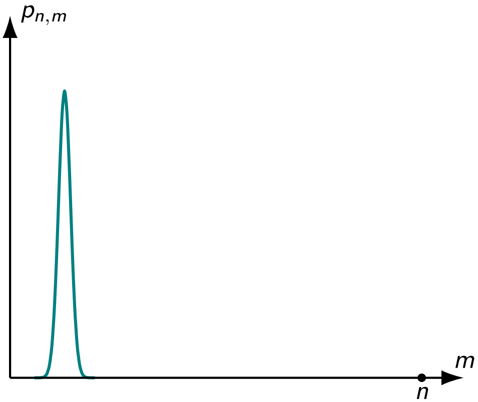


A illustration of the p.d.f. $p_{n,m}$.

The p.d.f. of $\mathcal{N}_n$: LLN & CLT



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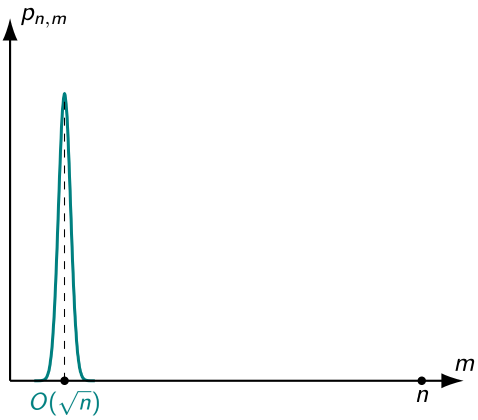
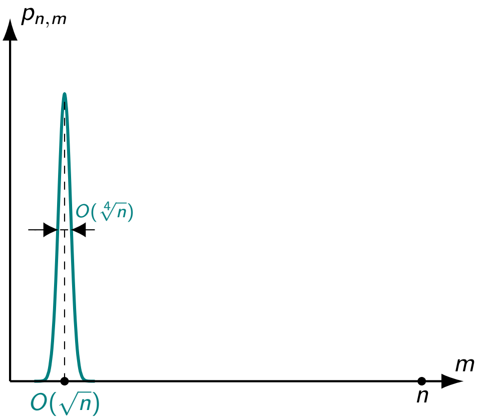


The Law of Large Number:

$$\mathbb{E}\mathcal{N}_n \sim \sqrt{\frac{2n}{\pi}}.$$

- Edelman–Kostlan–Shub '94, *J. Amer. Math. Soc.*



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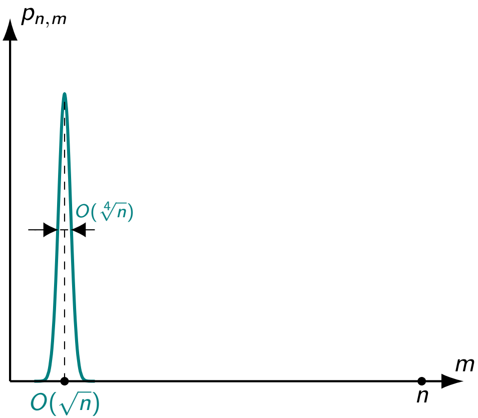
- Simm '17, *Random Matrices Theor. Appl.*
- Simm–FitzGerald '23, *Ann. Inst. Henri Poincaré Probab. Stat.*

The p.d.f. of $\mathcal{N}_n$: large deviation probabilities



The Rare Event (left tail):

$$\log p_{n,m} \sim -\frac{1}{\sqrt{2\pi}}\zeta(\tfrac{3}{2})\sqrt{n}, \qquad m = O(\tfrac{\mathbb{E}\mathcal{N}_n}{\log n}).$$

- KANZIEPER *et. al.* '16, *Ann. Appl. Probab.*

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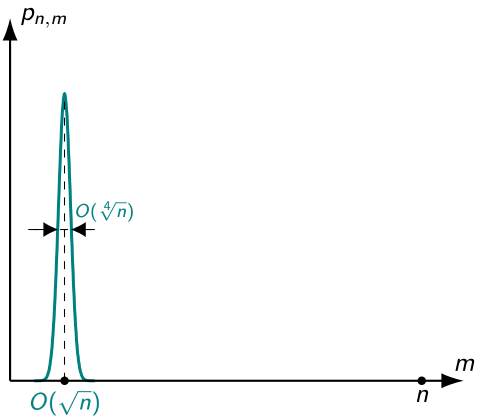

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- Kanzieper *et. al.* '16, *Ann. Appl. Probab.*

The Rare Event (right tail):

$$\log p_{n,m} \sim \begin{cases} a_1 n^2 + a_2 n \\ \quad + a_3 \log n + a_4 1, & m = n-2, \\ -\frac{\log 2}{4} n(n-1), & m = n. \end{cases}$$

- Edelman '97, *J. Multivariate Anal.*
- Akemann–Kanzieper '07, *J. Stat. Phys.*

Elliptic Ginibre Orthogonal Ensemble (elliptic GinOE)

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$$H_{+} = \frac{G + G^{T}}{\sqrt{2}}.$$

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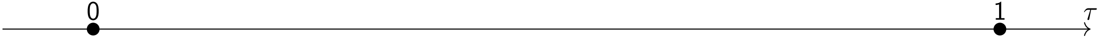

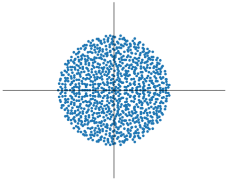

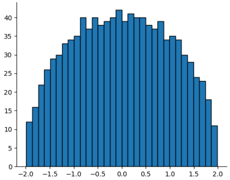

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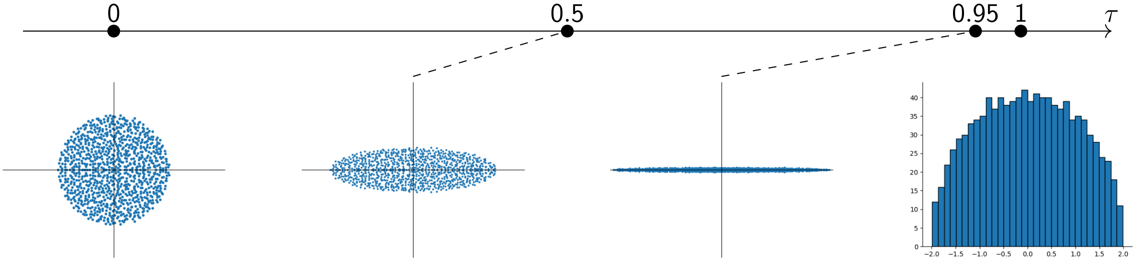

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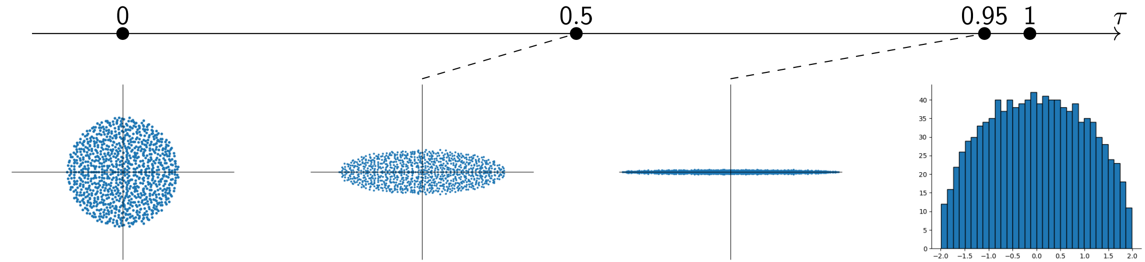
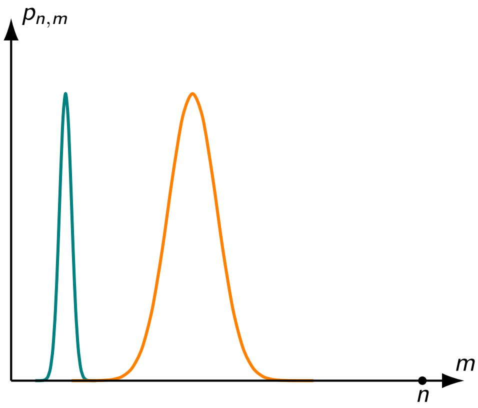


Strong non-Hermiticity: $\tau \in [0,1)$ is a constant. / **Weak non-Hermiticity**: $\tau \equiv \tau_n \uparrow 1$ as $n \to \infty$.

FYODOROV–KHORUZHENKO–SOMMERS '97&'98, *Phys. Rev. Lett.* & *Ann. Inst. H. Poincaré Phys. Théor.*

The p.d.f. of $\mathcal{N}_n$: LLN & CLT



- Strong non-Hermiticity: $\tau \in [0, 1)$: const.
- Weak non-Hermiticity: $\tau = 1 - \frac{\alpha^2}{n}$.

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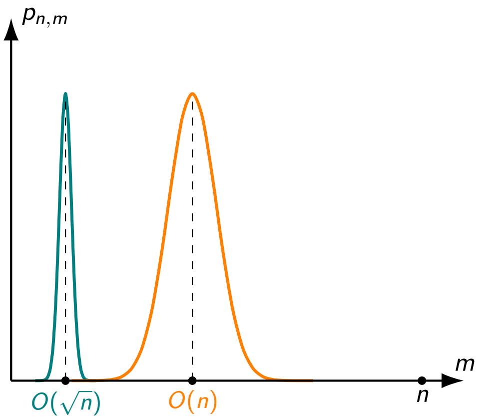

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$$\mathbb{E}\mathcal{N}_n \sim \begin{cases} \sqrt{\frac{2}{\pi}\frac{1+\tau}{1-\tau}}\sqrt{n}, & \text{Strong nH}, \\ c(\alpha)n, & \text{Weak nH}. \end{cases}$$

- Forrester–Nagao '08, *J. Phys. A.*
- Byun–Kang–Lee–Lee '23, *Int. Math. Res. Not.*

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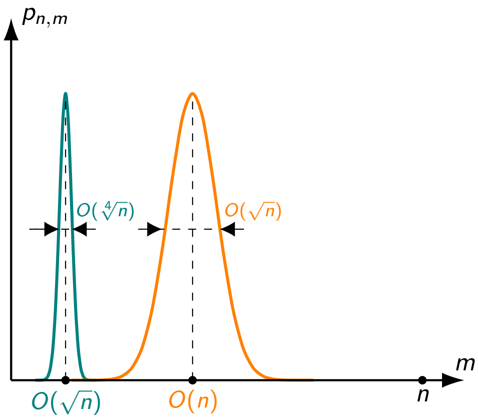

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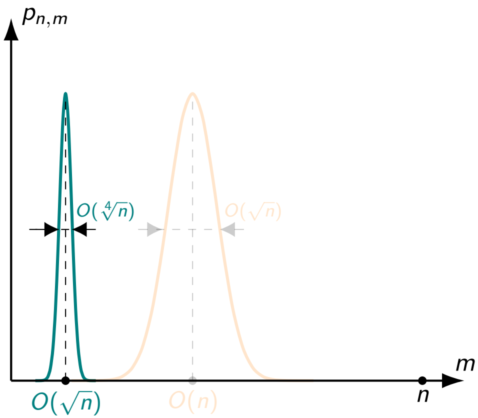
The Central Limit Theorem:

$$\frac{\mathcal{N}_n - \mathbb{E}\mathcal{N}_n}{\sqrt{\mathbb{E}\mathcal{N}_n}} \to \begin{cases} \mathrm{N}(0, 2-\sqrt{2}), & \text{Strong nH}, \\ \mathrm{N}(0, 2 - 2\frac{c(\sqrt{2}\alpha)}{c(\alpha)}), & \text{Weak nH}. \end{cases}$$

- Forrester '24, *Electron. Commun. Probab.*
- Byun–Molag–Simm '25, *Electron. J. Probab.*

The p.d.f. of $\mathcal{N}_n$: large deviation probabilities



The Rare Event (Strong nH):

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- Strong non-Hermiticity: $\tau \in [0,1)$: const.
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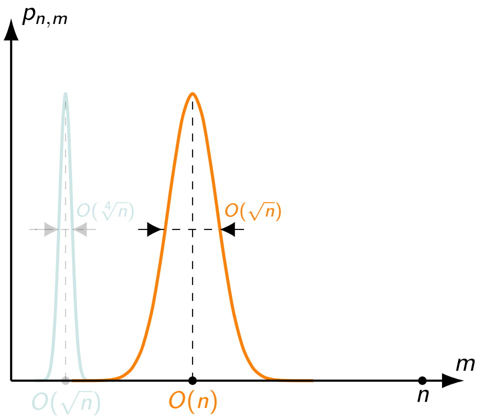
The p.d.f. of $\mathcal{N}_n$: large deviation



- Strong non-Hermiticity: $\tau \in [0,1)$: const.
- Weak non-Hermiticity: $\tau = 1 - \frac{\alpha^2}{n}$.

The Rare Event (Weak nH):

$$\log p_{n,m} = \begin{cases} \leq d(\alpha) n, & m = O\left(\frac{\mathbb{E}\mathcal{N}_n}{\log n}\right), \\ -\frac{n(n-1)}{4} \log\left(\frac{2}{1+\tau_n}\right), & m = n. \end{cases}$$

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Main result: elliptic GinOE

Theorem (Akemann–Byun–L. '25)

- **(General case)** *Let* $m = n - 2\ell$ *with* $\ell = O(1)$. *Then as* $n \to \infty$, *we have*

$$\log p_{n,n-2\ell} = \begin{cases} a_1 n^2 + a_2 n + a_3 \log n + O(1), & \textit{Strong nH}, \quad \textit{i.e.}, \quad \tau \in [0,1): \ \textit{const.}, \\ b_1 n + b_2 \log n + b_3 1 + o(1), & \textit{Weak nH}, \quad \textit{i.e.}, \quad \tau = 1 - \alpha^2/n. \end{cases}$$

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Here, the constants a_j and b_j are given by

$$a_1 = -\frac{1}{4}\log\Big(\frac{2}{1+\tau}\Big), \qquad a_2 = \ell \log\Big(\frac{3-\tau}{1+\tau}\Big) + \frac{1}{4}\log\Big(\frac{2}{1+\tau}\Big), \qquad a_3 = -\frac{\ell^2}{2},$$

$$b_1 = -\frac{\alpha^2}{8}, \qquad b_2 = \ell, \qquad b_3 = \frac{\alpha^2}{8} - \frac{\alpha^4}{32} + \frac{\ell}{2} e^{\frac{\alpha^2}{2}} (I_0(\tfrac{\alpha^2}{2}) - I_1(\tfrac{\alpha^2}{2})) - \ell - \log(\ell!).$$

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- **(Special case)** *Moreover, for $\ell = 1$, we have the full-order expansion.*

Remarks

- Equivalently, for $\ell = O(1)$, we have

$$\frac{p_{n,n-2\ell}}{p_{n,n}} \approx \begin{cases} \dfrac{1}{n^{\ell(\ell-1)/2}} \left(\dfrac{p_{n,n-2}}{p_{n,n}} \right)^{\ell} & \text{Strong nH.} \\[2ex] \dfrac{1}{\ell!} \left(\dfrac{p_{n,n-2}}{p_{n,n}} \right)^{\ell} & \text{Weak nH.} \end{cases}$$

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- Recall for $\tau = 1 - \alpha^2/n$, we have $\mathbb{E}\mathcal{N}_n = c(\alpha)n + O(1)$. Especially, we have

$$\frac{e^{\alpha^2/2}}{2}\left(I_0(\tfrac{\alpha^2}{2}) - I_1(\tfrac{\alpha^2}{2})\right) - 1 = \frac{c(i\alpha)}{2} - 1.$$

Further related topics

Real eigenvalue statistics

- Products of GinOE: Forrester '14, Forrester–Ipsen '16, Simm '17, Akemann–Byun '24
- Asymmetric Wishart matrix: Akemann–Kieburg–Phillips '10, Byun–Noda '25
- TOE: Forrester–Ipsen '18, Forrester–Ipsen–Kumar '20, Little–Mezzadri–Simm '22
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- (Elliptic) Ginibre ensembles: Amkemann–Byun–Ebke–Schehr '22, Akemann–Duits–Molag '24
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Relavent physical model

- Annihilating Brownian motion: Tribe–Zaboronski '11, Forrester '15

Studies on $\mathcal{N}_n$: Summary

LLN and CLT for $\mathcal{N}_n$

Regime	LLN	CLT
Strong nH.	EKS'94 ($\tau = 0$), FN'08	SI'17 ($\tau = 0$), FO'24, BMS'25
Weak nH.	BKLL'23	FO'24, BMS'25

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Sketch of Proof

1. The partial j.p.d.f.
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The partial j.p.d.f. of the eigenvalues

The partial j.p.d.f. having m real eigenvalues and ℓ complex conj. pairs is

$$\mathcal{P}_{m,\ell}(\lambda_1,\dots,\lambda_m;z_1,\dots,z_\ell)=\frac{1}{Z_{m,\ell}}\prod_{1\le j<k\le m}|\lambda_j-\lambda_k|\prod_{j=1}^{m}\exp\left(-\tfrac{n\lambda_j^2}{2(1+\tau)}\right)\prod_{j=1}^{m}\prod_{k=1}^{\ell}(\lambda_j-z_k)(\lambda_j-\overline{z}_k)$$

$$\times\prod_{1\le j<k\le \ell}|z_j-z_k|^2|z_j-\overline{z}_k|^2\prod_{j=1}^{\ell}|z_j-\overline{z}_j|\operatorname{erfc}\left(\tfrac{\sqrt{n}|z_j-\overline{z}_j|}{\sqrt{2(1-\tau^2)}}\right)\exp\left(-\tfrac{n(z_j^2+\overline{z}_j^2)}{2(1+\tau)}\right)$$

- Lehmann–Sommers '91, *Phys. Rev. Lett.*, Forrester–Nagao '08, *J. Phys. A.*

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Our task

Find asymptotic formula for

$$p_{n,m}=\int_{\mathbb{R}^m}\int_{\mathbb{H}^\ell}\mathcal{P}_{m,\ell}(\lambda_1,\dots,\lambda_m;z_1,\dots,z_\ell)\,d^2z_1\cdots d^2z_\ell\,d\lambda_1\cdots d\lambda_m.$$

Two different approaches

1. Two-species 2d Coulomb gas:

$$\mathcal{P}_{m,\ell}(\lambda_1,\dots,\lambda_m;z_1,\dots,z_\ell)=\frac{1}{Z_{m,\ell}}\prod_{1\le j<k\le m}|\lambda_j-\lambda_k|\prod_{j=1}^{m}\exp\left(-\frac{n\lambda_j^2}{2(1+\tau)}\right)\prod_{j=1}^{m}\prod_{k=1}^{\ell}(\lambda_j-z_k)(\lambda_j-\overline{z}_k)$$

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Interpretation on the partial j.p.d.f.

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$$\mathcal{P}_{m,\ell}(\lambda_1,\dots,\lambda_m;z_1,\dots,z_\ell)=\frac{1}{Z_{m,\ell}}\mathbb{P}_{\mathrm{GOE}(m)}(G)\prod_{j=1}^{\ell}\det(z_j-G)\det(\bar{z}_j-G)$$

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Mean-field approximation

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Strong Szegő Theorem (Johansson '98, *Duke. Math. J.*, Shcherbina '13, *J. Stat. Phys.*)

For a test function $h : \mathbb{R} \to \mathbb{R}$ (with enough regularity), as $m \to \infty$,

$$\mathbb{E}_{\mathrm{GOE}(m)}\left[\exp\left(\sum_{j=1}^{m} h(\lambda_j)\right)\right] = \exp\left(m\int_{\mathbb{R}} h(\lambda)\,dg(\lambda) + A[h] + O(m^{-1})\right)$$

for some explicit functional A.

Apply the *Strong Szegő Theorem* with $h(\lambda) = \sum_{k=1}^{\ell} \log|\lambda - z_k|^2$.

Mean-field approximation and effective potential

$$p_{n,m} = \frac{1}{Z_{m,\ell}} \int_{\mathbb{H}^\ell} \overbrace{\int_{\mathbb{R}^m} \prod_{1\le j<k\le m} |\lambda_j - \lambda_k| \prod_{j=1}^{m} \exp\Big(-\frac{n\lambda_j^2}{2(1+\tau)}\Big) \prod_{j=1}^{m}\prod_{k=1}^{\ell} (\lambda_j - z_k)(\lambda_j - \bar{z}_k)\, d\vec{\lambda}}^{\mathbb{E}_{GOE}\left[\exp(\sum_j h(\lambda_j))\right] =}$$

$$\times \prod_{1\le j<k\le \ell} |z_j - z_k|^2 |z_j - \bar{z}_k|^2 \prod_{j=1}^{\ell} |z_j - \bar{z}_j| \operatorname{erfc}\Big(\frac{\sqrt{n}|z_j - \bar{z}_j|}{\sqrt{2(1-\tau^2)}}\Big) \exp\Big(-\frac{n(z_j^2 + \bar{z}_j^2)}{2(1+\tau)}\Big)\, d^2\vec{z}$$

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$$\overbrace{}^{\mathbb{E}_{GOE}\left[\exp(\sum_j h(\lambda_j))\right]=}$$

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$$= \frac{Z_{GOE(m)}}{Z_{m,\ell}} \int_{\mathbb{H}^\ell} e^{m \int_{\mathbb{R}} \sum_j \log|\lambda - z_j|^2 \, d\mu_{sc}(\lambda) + O(1)} \prod_{1\le j<k\le \ell} |z_j - z_k|^2 |z_j - \bar{z}_k|^2 \prod_{j=1}^{\ell} e^{-nV_n(z_j)}\, d^2\vec{z}$$

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where

$$Q_n(z) = V_n(z) + \frac{m}{n} \int_{\mathbb{R}} \log|z-\lambda|^2\, d\mu_{sc}(\lambda) \xrightarrow{n\to\infty} Q(z).$$

Effective potential: strong non-Hermiticity

$$Q(z) = \frac{(\mathrm{Re}\, z)^2}{1+\tau} + \frac{(\mathrm{Im}\, z)^2}{1-\tau} - \int_{\mathbb{R}} \log |z-t|^2 \, d\mu_{sc}(t).$$

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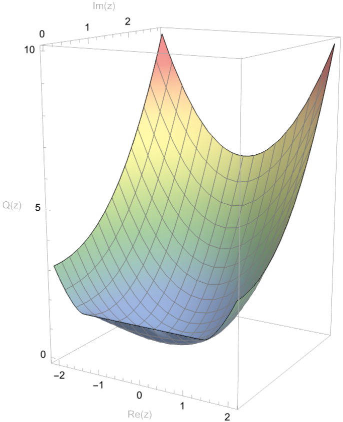

Figure: $Q(z)$ for $z \in \mathbb{H}$.

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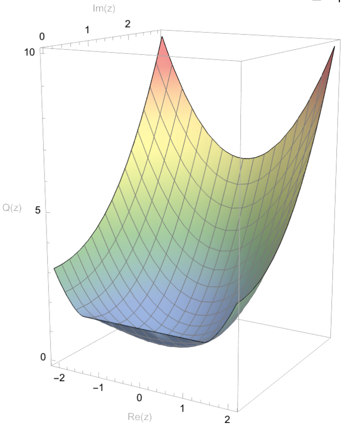

Figure: $Q(z)$ for $z \in \mathbb{H}$.

- $\exists$ the global minima $z^\star := \operatorname{argmin}_{z\in\mathbb{H}} Q(z)$.

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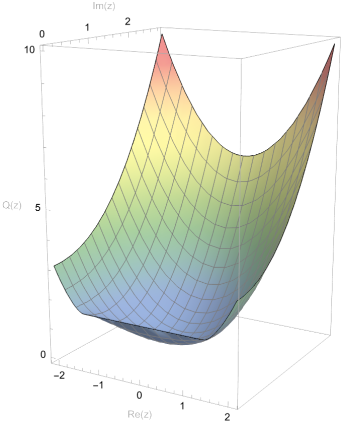

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$$Q(z^\star) - Q(0) = \frac{3-\tau}{1+\tau}.$$

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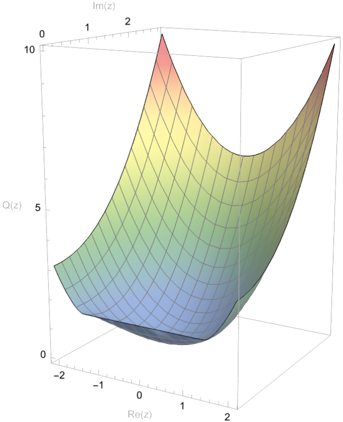

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- Heuristically, $|z_j - z_k| = O(1/\sqrt{n})$ for $j \neq k$. This implies

$$\prod_{1 \le j < k \le \ell} |z_j - z_k|^2 \asymp \frac{1}{n^{\ell(\ell-1)/2}}.$$

Conclusion of the proof

Recall that

$$p_{n,m} = \frac{Z_{GOE(m)}}{Z_{m,\ell}} \int_{\mathbb{H}^\ell} \prod_{1\le j<k\le \ell} |z_j - z_k|^2 |z_j - \overline{z}_k|^2 \prod_{j=1}^{\ell} e^{-nQ_n(z_j)}\, d^2\vec{z} e^{O(1)}.$$

Thus,

$$\frac{p_{n,n-2\ell}}{p_{n,n}} \approx \frac{1}{n^{\ell(\ell-1)/2}} \Big(\frac{p_{n,n-2\ell}}{p_{n,n}}\Big)^{\ell n} \approx \frac{1}{n^{\ell^2/2}} \Big(\frac{3-\tau}{1+\tau}\Big)^{\ell n}.$$

□

Effective potential: weak non-Hermiticity (Heuristic)

$$Q_n(z) \approx \frac{(\operatorname{Re} z)^2}{1+\tau_n} + \frac{(\operatorname{Im} z)^2}{1-\tau_n} - \int_{\mathbb{R}} \log|z-t|^2 \, d\mu_{sc}(t).$$

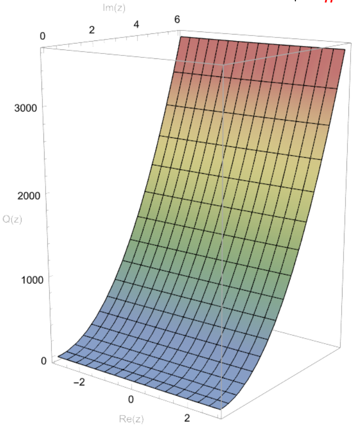

Figure: $Q_n(z)$ for $z \in \mathbb{H}$ ($\tau_n = 0.99$).

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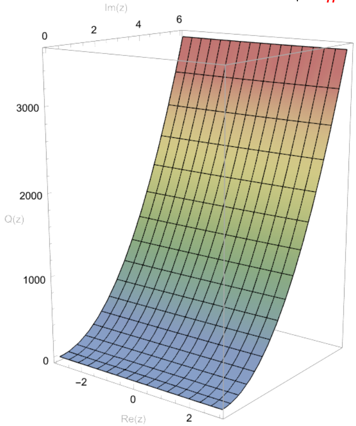

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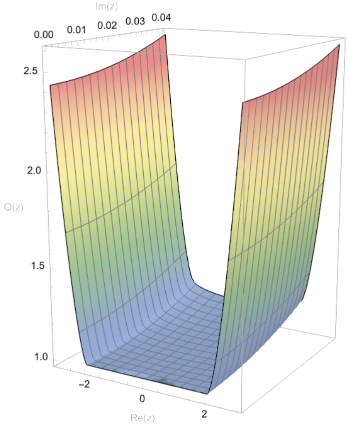

Figure: $Q_n(z)$ for $z \in \mathbb{H}$: zoomed.

Sketch of Proof

1. The partial j.p.d.f.
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Products of characteristic polynomials of the GOE

Recall that the partial j.p.d.f.

$$\mathcal{P}_{m,\ell}(\lambda_1,\dots,\lambda_m; z_1,\dots,z_\ell) = \frac{1}{Z_{m,\ell}} \mathbb{P}_{\mathrm{GOE}(m)}(G) \prod_{j=1}^{\ell} \det(z_j - G)\det(\overline{z}_j - G) \times \underbrace{\prod_{1\le j<k\le \ell} |z_j - z_k|^2 |z_j - \overline{z}_k|^2 \prod_{j=1}^{\ell} |z_j - \overline{z}_j| \exp\left(-\frac{n(z_j^2+\overline{z}_j^2)}{2(1+\tau)}\right)}_{=:\Delta_w(\vec{z})} \operatorname{erfc}\left(\frac{\sqrt{n}|z_j-\overline{z}_j|}{\sqrt{2(1-\tau^2)}}\right).$$

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$$\times\underbrace{\prod_{1\le j<k\le\ell}|z_j-z_k|^2|z_j-\overline{z}_k|^2\prod_{j=1}^{\ell}|z_j-\overline{z}_j|\exp\left(-\frac{n(z_j^2+\overline{z}_j^2)}{2(1+\tau)}\right)}_{=:\Delta_w(\vec{z})}\mathrm{erfc}\left(\frac{\sqrt{n}|z_j-\overline{z}_j|}{\sqrt{2(1-\tau^2)}}\right).$$

Especially, it is known

$$\left\langle\prod_{j=1}^{\ell}\det(z_j-G)\det(\overline{z}_j-G)\right\rangle_{\mathrm{GOE}(m)}=\frac{1}{\Delta_w(\vec{z})}\mathrm{Pf}\begin{bmatrix}\kappa_n(z_a,z_b) & \kappa_n(z_a,\overline{z}_b)\\ \kappa_n(\overline{z}_a,z_b) & \kappa_n(\overline{z}_a,\overline{z}_b)\end{bmatrix}_{2\ell\times2\ell},$$

Products of characteristic polynomials of the GOE

Recall that the partial j.p.d.f.

$$\mathcal{P}_{m,\ell}(\lambda_1,\dots,\lambda_m;z_1,\dots,z_\ell)=\frac{1}{Z_{m,\ell}}\mathbb{P}_{\mathrm{GOE}(m)}(G)\prod_{j=1}^{\ell}\det(z_j-G)\det(\overline{z}_j-G)$$
$$\times\underbrace{\prod_{1\le j<k\le\ell}|z_j-z_k|^2|z_j-\overline{z}_k|^2\prod_{j=1}^{\ell}|z_j-\overline{z}_j|\exp\Big(-\frac{n(z_j^2+\overline{z}_j^2)}{2(1+\tau)}\Big)}_{=:\Delta_w(\vec{z})}\operatorname{erfc}\Big(\frac{\sqrt{n}|z_j-\overline{z}_j|}{\sqrt{2(1-\tau^2)}}\Big).$$

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where $\kappa_n(z,w)$ is the GOE skew-kernel

$$\kappa_n(z,w)=\frac{1}{2}e^{-\frac{z^2+w^2}{2}}\sum_{j=0}^{n/2-1}\frac{q_{2j+1}(z)q_{2j}(w)-q_{2j}(z)q_{2j+1}(w)}{h_j}.$$

- Borodin–Strahov '06, *Comm. Pure Appl. Math.*

Pfaffian formula

These give

$$p_{n,m} = \frac{p_{n,n}}{\ell!}\left(\frac{2}{i}\right)^{\ell}\int_{\mathbb{H}^{\ell}} \mathrm{Pf}\begin{bmatrix} \kappa_n(z_a, z_b) & \kappa_n(z_a, \overline{z}_b) \\ \kappa_n(\overline{z}_a, z_b) & \kappa_n(\overline{z}_a, \overline{z}_b) \end{bmatrix}_{2\ell\times 2\ell} \prod_{j=1}^{\ell} \mathrm{erfc}\left(\frac{|z_j - \overline{z}_j|}{\sqrt{2(1-\tau)}}\right) d^2\vec{z}.$$

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The summation formula for the GOE skew-kernel is

$$\begin{aligned} \kappa_n(z,w) &= \frac{1}{2} e^{-\frac{z^2+w^2}{2}} \sum_{j=0}^{n/2-1} \frac{q_{2j+1}(z)q_{2j}(w) - q_{2j}(z)q_{2j+1}(w)}{h_j} \\ &= e^{\frac{z^2+w^2}{2}} \frac{c_n}{c_{n-1}} \left[\frac{d}{dz}\left(\frac{\psi_n(z)\psi_{n-1}(w) - \psi_{n-1}(z)\psi_n(w)}{z-w}\right) + \psi_n(z)\psi_{n-1}(w)\right], \end{aligned}$$

where $\psi_n(z)$ is the Hermite function and c_n's are explicit constants.

- Forrester–Nagao–Honner '99, *Nuclear Phys. B.*, Widom '99, *J. Stat. Phys.*, Adler–Forrester–Nagao–van Moerbeke '00 *J. Stat. Phys.*

Asymptotic of the Pfaffian: weak non-Hermiticity

For general $\ell = O(1)$,

$$\mathrm{Pf}\begin{bmatrix} \kappa_n(z_a, z_b) & \kappa_n(z_a, \overline{z}_b) \\ \kappa_n(\overline{z}_a, z_b) & \kappa_n(\overline{z}_a, \overline{z}_b) \end{bmatrix}_{2\ell\times 2\ell} \approx \prod_{j=1}^{\ell} \kappa_n(z_j, \overline{z}_j).$$

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Thus, we have

$$p_{n,n-2\ell} = \frac{p_{n,n}}{\ell!}\left(\frac{2}{i}\right)^{\ell} \int_{\mathbb{H}^\ell} \mathrm{Pf}\begin{bmatrix} \kappa_n(z_a, z_b) & \kappa_n(z_a, \overline{z}_b) \\ \kappa_n(\overline{z}_a, z_b) & \kappa_n(\overline{z}_a, \overline{z}_b) \end{bmatrix}_{2\ell\times 2\ell} \prod_{j=1}^{\ell} \mathrm{erfc}\left(\frac{z_j - \overline{z}_j}{i\sqrt{2(1-\tau)}}\right) d^2\vec{z}$$

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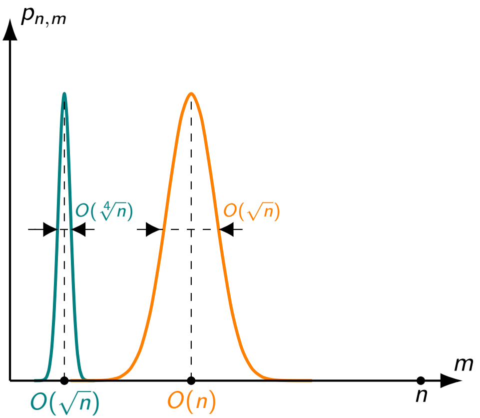
□

Summary

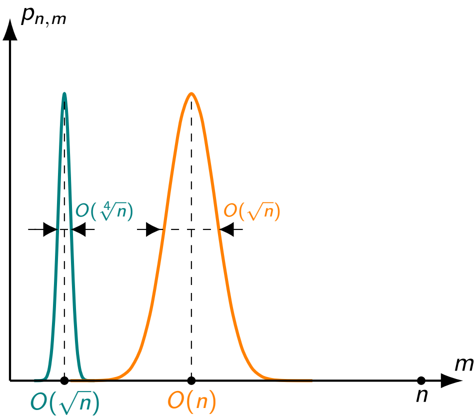


- Strong non-Hermiticity: $\tau \in [0,1)$: const.
- Weak non-Hermiticity: $\tau = 1 - \frac{\alpha^2}{n}$.

What we have done

Regime	$m = O(1)$	$m = n - O(1)$	$m = n$
Strong nH.	KPTTZ '16 BMS '25	AK'07 $(m = n - 2)$	Ed '97, FN '08 (closed form)
Weak nH.	BMS '25 (upper bound)		FN '08 (closed form)

Summary



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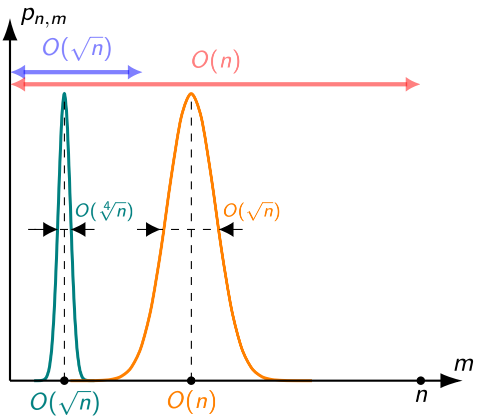

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Work in progress

- Moderate deviation for $\mathcal{N}_n$.
  - with S.-S. Byun, J. Jalowy & G. Schehr.

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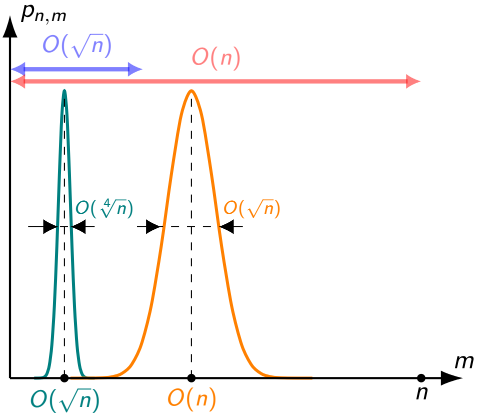

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- Moderate deviation for $\mathcal{N}_n$.
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- Fluctuation of $\mathcal{N}_n$ for $O(n)$-product of GinOEs.
  - with S.-S. Byun & K. Noda.

Thank you for your attention!