Riemann Hilbert problems

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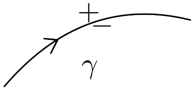
Outline of the talk

1. **Riemann Hilbert problems**
2. **Steepest descent analysis**
3. **Non-hermitian orthogonality and S-curves**
4. **Multiple orthogonal polynomials**
5. **Matrix valued orthogonal polynoials**

1. Riemann Hilbert problems

Riemann Hilbert problem

- **Jump problem for a piecewise analytic function**



Scalar RH problem (additive jump):

RH-f1 $f : \mathbb{C} \setminus \Gamma \to \mathbb{C}$ **is analytic**

RH-f2 f **has boundary values on both sides of** Γ**, and**

$$f_+ = f_- + v \qquad \textbf{on } \Gamma$$

RH-f3 $f(z) = \mathcal{O}(z^{-1})$ **as** $z \to \infty$**.**

Unique solution is

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{v(s)}{s - z} ds.$$

Matrix Riemann-Hilbert problem for OPs

Given weight w on $\mathbb{R}$ and $n \in \mathbb{N}$, find 2 × 2 matrix valued function $Y(z)$ such that

RH-Y1 $Y : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{2\times 2}$ **is analytic.**

RH-Y2 Y **has boundary values on $\mathbb{R}$, and and**

$$Y_+ = Y_- \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}, \quad \textbf{on } \mathbb{R}.$$

RH-Y3 $Y(z) = (I + \mathcal{O}(z^{-1})) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}$ **as $z \to \infty$.**

Unique solution is given in terms of orthogonal polynomials

$$\int_{-\infty}^{\infty} P_n(x) x^k w(x) dx = 0 \quad k = 0, 1, \ldots, n-1,$$

Fokas, Its, Kitaev RH problem for OP

RH problem has the unique solution

$$Y(z) = \begin{pmatrix} P_n(z) & \dfrac{1}{2\pi i}\displaystyle\int_{-\infty}^{\infty} \dfrac{P_n(s)w(s)}{s-z}ds \\ -2\pi i\gamma_{n-1}^{-2} P_{n-1}(z) & -\gamma_{n-1}^{-2}\displaystyle\int_{-\infty}^{\infty} \dfrac{P_{n-1}(s)w(s)}{s-z}ds \end{pmatrix}$$

where γ_{n-1} is the leading coefficient of the orthonormal polynomial of degree $n-1$.

Fokas-Its-Kitaev (1992)

2. Steepest descent analysis

Deift-Zhou steepest descent analysis

Deift-Zhou (1993) steepest descent analysis for RH problem for A on contour Σ_A, depending on parameter, say n, and we are interested in $n \to \infty$.

- **Sequence of explicit transformations** $A \mapsto B \mapsto \cdots \mapsto R$ **leading to a RH problem for R on contour Σ_R**

RH-R1 $R : \mathbb{C} \setminus \Sigma_R \to \mathbb{C}^{2\times 2}$ **is analytic.**

RH-R2 R **has boundary values on Σ_R satisfying**

$$R_+ = R_- J_R, \quad \text{on } \Sigma_R,$$

where J_R depends on n with $\boxed{J_R \to I}$ **as $n \to \infty$, both in $L^2(\Sigma_R)$ and $L^\infty(\Sigma_R)$.**

RH-R3 $R(z) = I + \mathcal{O}(z^{-1})$ **as $z \to \infty$.**

As a result

$$\boxed{R(z) \to I \quad \text{as } n \to \infty}$$

uniformly for z in compact subsets of $\mathbb{C} \setminus \Sigma_R$.

RH problem with varying exponential weight

We apply Deift/Zhou steepest descent analysis to following
Deift-Kriecherbauer-McLaughlin-Venakides-Zhou (1999)

RH-Y1 $Y : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{2\times 2}$ **is analytic,**

RH-Y2 $Y_+(x) = Y_-(x) \begin{pmatrix} 1 & e^{-nV(x)} \\ 0 & 1 \end{pmatrix}$ **on** $\mathbb{R}$**,**

RH-Y3 $Y(z) = \left(I + \mathcal{O}\left(z^{-1}\right)\right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}$ **as** $z \to \infty$**.**

Sequence of transformations

$$\boxed{Y \mapsto T \mapsto S \mapsto R}$$

$Y \mapsto T$ **: normalization by means of the equilibrium measure**

$T \mapsto S$ **: opening of lenses, turning oscillations into exponentially decaying entries**

$S \mapsto R$ **: construction of global and local parametrices**

Equilibrium measure in external field

μ_V is the probability measure that minimizes the logarithmic energy in the external field

$$\iint \log \frac{1}{|x-y|} d\mu(x) d\mu(y) + \int V(x) d\mu(x)$$

- For real analytic V, there is a density ψ_V which is supported on a finite union of intervals, and that is real analytic on the interior of each interval.

Deift-Kriecherbauer-McLaughlin (1999)

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Deift-Kriecherbauer-McLaughlin (1999)

In many situations it is important that the Cauchy transform of μ_V satisfies quadratic equation

$$\left[\int \frac{d\mu_V(s)}{s-z} + \frac{V'(z)}{2}\right]^2 = \underbrace{\left(\frac{V'(z)}{2}\right)^2 - \int \frac{V'(z)-V'(s)}{z-s} d\mu_V(s)}_{=Q_V(z)}$$

and Q_V is a polynomial in case V is a polynomial.

First transformation $Y \mapsto X$

We use g-function

$$g(z) = \int \log(z - s) d\mu_V(s)$$

Define for suitable constant ℓ,

$$T(z) = \begin{pmatrix} e^{n\ell/2} & 0 \\ 0 & e^{-n\ell/2} \end{pmatrix} Y(z) \begin{pmatrix} e^{-ng(z)-n\ell/2} & 0 \\ 0 & e^{ng(z)+n\ell/2} \end{pmatrix}$$

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T satisfies a new RH problem. Jumps take different shape on the support of μ_V and outside the support.

- **Suppose** $\mathrm{supp}(\mu_V) = \bigcup_{j=1}^{N} [a_{2j-1}, a_{2j}]$.
- **Each end-point has φ-function**

$$\varphi_j(z) = \int_{a_j}^{z} Q_V(s)^{1/2} ds$$

RH problem for T

T **satisfies**

RH-T1 $T : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^2$ **is analytic,**

RH-T2 T **has boundary values on** $\mathbb{R}$ **satisfying**

RH-T2a $T_+ = T_- \begin{pmatrix} e^{2n\varphi_{2N,+}} & 1 \\ 0 & e^{2n\varphi_{2N,-}} \end{pmatrix}$ **on** $\operatorname{supp} \mu_V$**,**

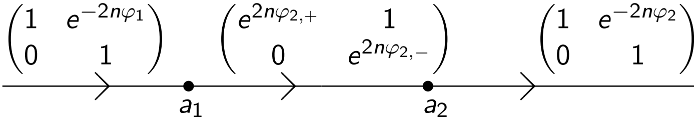
RH-T2b $T_+ = T_- \begin{pmatrix} 1 & e^{-2n\varphi_1 O} \\ 0 & 1 \end{pmatrix}$ **on** $(-\infty, a_1)$**.**

RH-T2c $T_+ = T_- \begin{pmatrix} e^{-2\pi i n \omega_j} & e^{-2n\varphi_{2j}} \\ 0 & e^{2\pi i n \omega_j} \end{pmatrix}$ **on**(a_{2j}, a_{2j+1}) **for** $j = 1, \ldots, N-1$, **with** $\omega_j = \mu_V([a_{2j+1}, \infty))$,

RH-T2d $T_+ = T_- \begin{pmatrix} 1 & e^{-2n\varphi_{2N}} \\ 0 & 1 \end{pmatrix}$ **on** (a_{2N}, ∞)**,**

RH-T3 $\boxed{T(z) = I + \mathcal{O}(z^{-1})}$ **as** $z \to \infty$**.** **(normalization)**

One interval case



- $\varphi_1(x) > 0$ **for** $x < a_1$**,**
- $\varphi_2(x) > 0$ **for** $x > a_2$**,**
- $\varphi_{2,+} = -\varphi_{2,-}$ **is purely imaginary on** (a_1, a_2)

Second transformation $T \mapsto S$

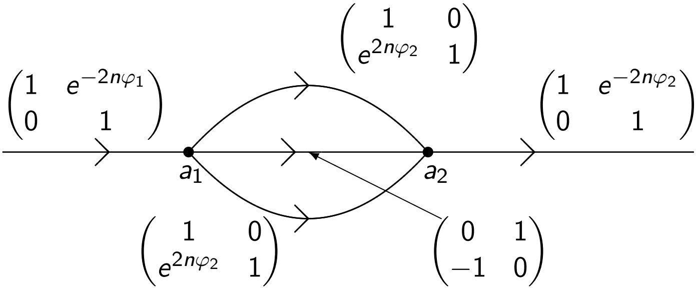
Open a lens around $[a_1, a_2]$ and define

$$S = T \begin{pmatrix} 1 & 0 \\ -e^{2n\varphi_2} & 1 \end{pmatrix} \quad \textbf{in upper part of the lens}$$

$$S = T \begin{pmatrix} 1 & 0 \\ e^{2n\varphi_2} & 1 \end{pmatrix} \quad \textbf{in lower part of the lens}$$

$S = T$ **outside the lens**

Jumps in the RH problem for S



Global parametrix

Parametrices are approximations to S.

Away from endpoints S is well-approximated by the global parametrix M: it is the solution to the RH problem where we forget about the non-constant jumps

RH-M1 $M : \mathbb{C} \setminus [a_1, a_2] \to \mathbb{C}^{2\times 2}$ **is analytic.**

RH-M2 $M_+ = M_- \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ **on** (a_1, a_2)**.**

RH-M3 $M(z) = I + \mathcal{O}\left(z^{-1}\right)$ **as** $z \to \infty$**.**

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RH-M3 $M(z) = I + \mathcal{O}\left(z^{-1}\right)$ as $z \to \infty$.

The solution is explicit (with $\beta(z) = \left(\frac{z-a_2}{z-a_1}\right)^{1/4}$)

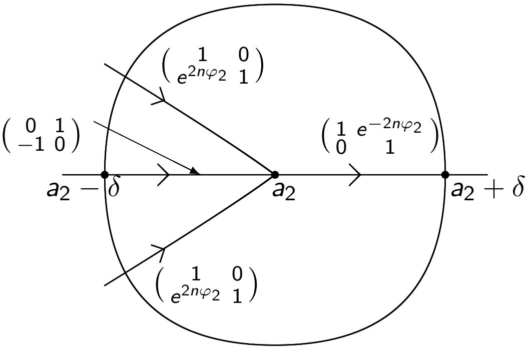
$$M(z) = \begin{pmatrix} \frac{1}{2}\left(\beta(z) + \beta^{-1}(z)\right) & \frac{1}{2i}\left(\beta(z) - \beta^{-1}(z)\right) \\ -\frac{1}{2i}\left(\beta(z) - \beta^{-1}(z)\right) & \frac{1}{2}\left(\beta(z) + \beta^{-1}(z)\right) \end{pmatrix}$$

Global parametrix is more complicated in multi-interval case.

Local parametrix

Local parametrix P **approximates** S **near endpoints** a_1, a_2.

Jump matrices for P **near** a_2 **agree with those of** S



Matching condition:

$$P(z) = \left(I + O\left(n^{-1}\right)\right) M(z)$$

as $n \to \infty$**, uniformly for** $|z - a_2| = \delta$

P **is constructed with Airy functions in case the density** ψ_V **of the equilibrium measure vanishes as a square root at** a_2.

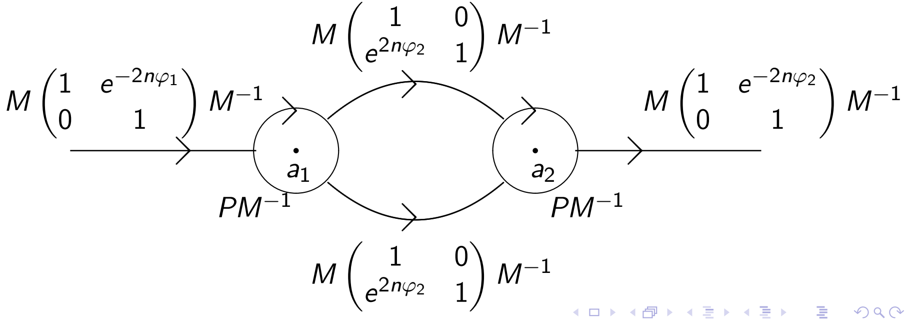
Third transformation $S \mapsto R$

Define $$R(z) = \begin{cases} S(z)M(z)^{-1}, & \text{for } z \text{ outside disks}, \\ S(z)P(z)^{-1}, & \text{for } z \text{ inside disks} \end{cases}$$

RH problem for R

- RH-R1 $R : \mathbb{C} \setminus \Sigma_R \to \mathbb{C}^{2\times 2}$ **is analytic.**
- RH-R2 $R_+ = R_- J_R$ **on** Σ_R.
- RH-R3 $R(z) = I + \mathcal{O}(z^{-1})$ **as** $z \to \infty$.

Jumps for R are all $I + \mathcal{O}(n^{-1})$ as $n \to \infty$



3. Non-hermitian orthogonality and S-contours

Non hermitian orthogonality on a contour

In several contexts one is interested in polynomials P_n satisfying

$$\int_\Gamma P_n(z) z^k e^{-nV(z)} dz = 0, \qquad k = 0, 1, \ldots, n-1,$$

- Γ is a contour in the complex plane
- $V(z)$ is holomorphic in domain Ω containing Γ

Good news

- RH problem and its solution remain valid.
- Equilibrium measure with external field V on Γ exists.

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Good news

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Complication

- **Γ is not unique: We can move Γ within Ω by Cauchy's theorem. Each Γ has its own equilibrium measure.**
- **Is there a "good" contour Γ ?**

S property

During the RH analysis one uses the g-function

$$g(z) = \int \log(z-s) d\mu_V(s)$$

where μ_V now also depends on Γ.

- Important property

$$\mathrm{Re}\,(g_+(z) + g_-(z) - V(z)) = \ell \qquad \text{on support of } \mu_V$$

To make the steepest descent analysis work one also needs that the imaginary part is constant on each component of the support. This is (equivalent to) the S-property

$$\mathrm{Im}\,(g_+(z) + g_-(z) - V(z)) \quad \text{is piecewise constant on support of } \mu_V$$

- The support of the equilibrium measure should be an S-contour in external field V.

Example: Ginibre ensemble with insertion

Planar orthogonal polynomials (POP) appear in the analysis of normal matrix model and other random matrix models with eigenvalues in the complex plane.

- **POP are sometimes non-hermitian orthogonal on a contour.**
- **Example: Ginibre ensemble with insertion at** $a > 0$

$$\int_{\mathbb{C}} P_n(z)\overline{z}^k |z-a|^{2nc} e^{-n|z|^2} dA(z) = 0, \qquad k = 0, 1, \ldots, n-1$$

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$$\int_{\mathbb{C}} P_n(z)\bar{z}^k |z-a|^{2nc} e^{-n|z|^2} dA(z) = 0, \qquad k = 0, 1, \dots, n-1$$

Same polynomial P_n **also satisfies**

$$\frac{1}{2\pi i} \oint_{\Gamma} P_n(z) z^k \frac{(z-a)^{cn}}{z^{cn+n}} e^{-anz} dz = 0, \qquad k = 0, 1, \dots, n-1,$$

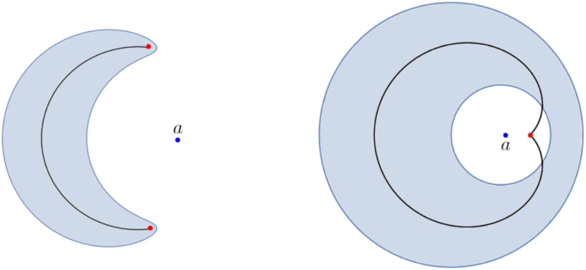
where Γ **is a closed contour around the interval** $[0, a]$.

Balogh, Bertola, Lee, McLaughlin (2015)

Phase transition

Model has a phase transition:

- **For large $c > 0$, the S-contour is an open arc.**
- **For small c, the S-contour is a closed contour.**



Analysis of critical case: Krüger, Lee, Yang (2025)

4. Multiple orthogonal polynomials

Multiple orthogonality of type II

Given

- **contour** Γ **with weights** $w_1, \ldots, w_r$
- **multi-index** $\vec{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r$

Type II MOP $P_{\vec{n}}$ **of degree** $|\vec{n}| = \sum_j n_j$ **satisfies**

$$\int_\Gamma P_{\vec{n}}(z) z^k w_j(z) dz = 0, \quad k = 0, \ldots, n_j - 1, \; j = 1, \ldots, r.$$

Riemann-Hilbert problem for MOP

RH problem has size $(r+1)\times(r+1)$

RH-Y1 $Y : \mathbb{C}\setminus\Gamma \to \mathbb{C}^{(r+1)\times(r+1)}$ **is analytic**

RH-Y2 Y **has boundary values on Γ that satisfy**

$$Y_+ = Y_- \begin{pmatrix} 1 & w_1 & \cdots & w_r \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \quad \textbf{on } \Gamma,$$

RH-Y3 $Y(z) = (I + \mathcal{O}(z^{-1})) \begin{pmatrix} z^{|\vec{n}|} & 0 & \cdots & 0 \\ 0 & z^{-n_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & z^{-n_r} \end{pmatrix}$ **as $z \to \infty$.**

Riemann-Hilbert problem for MOP

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RH-Y3 $Y(z) = (I+\mathcal{O}(z^{-1})) \begin{pmatrix} z^{|\vec{n}|} & 0 & \cdots & 0 \\ 0 & z^{-n_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & z^{-n_r} \end{pmatrix}$ **as $z\to\infty$.**

RH problem has a solution if and only if the type II MOP uniquely exists and in that case

$$\boxed{P_{\vec{n}}(z) = Y_{11}(z)}$$

Van Assche, Geronimo, K (2001)

MOP in RMT

MOP appear in a number of random matrix models where eigenvalues are a determinantal point process

- **Random matrices with external source**
- **Couples random matrices (two matrix model)**
- **Muttalib-Borodin ensemble**
- **Ginibre ensemble with more than one insertion**

Lee-Yang (2019)

Vector equilibrium problem

Asymptotic analysis of RH problem requires in many cases a vector of equilibrium measures

$$\vec{\mu} = (\mu_1, \ldots, \mu_r)$$

minimizing some energy functional

$$\sum_{j,k=1}^{r} c_{j,k} \iint \log \frac{1}{|z-w|} d\mu_j(z) d\mu_k(w) + \sum_{j=1}^{r} \int V_j d\mu_j$$

5. Matrix valued orthogonal polynomials

Matrix valued orthogonality

Given

- **Weight** W **is matrix valued of size** $r \times r$ **on** Γ

$P_n(z) = z_n I_r + \dots$ **is matrix valued polynomial of degree** n **if**

$$\int_\Gamma P_n(z) z^k W(z) dz = 0_r, \quad k = 0, 1, \dots, n-1$$

integration is done entrywise

Matrix valued orthogonality

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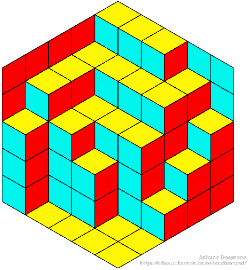
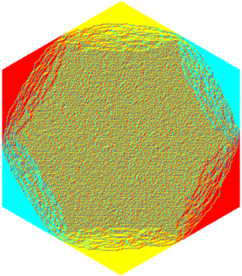
RH problem has a solution if and only if MVOP uniquely exists and in that case

$$\boxed{P_n(z) = \begin{pmatrix} I_r & 0_r \end{pmatrix} Y(z) \begin{pmatrix} I_r \\ 0_r \end{pmatrix}}$$

Random tilings

MVOP can be used in asymptotic analysis of random tilings with doubly periodic weightings **Duits, K (2021)**





One needs equilibrium measure in external field on a compact Riemann surface to normalize the RH problem

- **Steepest descent analysis of RH period is done for a special class of 3×3 periodic weightings** **K (2025)**
- **Implications for random tilings**

van Horssen-K (coming soon)

Thank you for your attention

Happy Birthday,
Peter !!