Phase Diagram of Extensive-Rank Matrix Denoising beyond Rotational Invariance

Justin Ko — University of Waterloo → Syracuse University

LiCA 2025 – Matrix Institute

Based on three joint papers with:

1. PRX 2025: Jean Barbier (ICTP), Francesco Camilli (ICTP), Koki Okajima (University of Tokyo)
2. IZS 2024: Jean Barbier (ICTP), Anas Rahman (Hong Kong University)
3. PTRF 2025: Jonathan Husson (Université Clermont-Auvergne)

August 10, 2025

Spiked Wigner Matrix: Johnstone '00

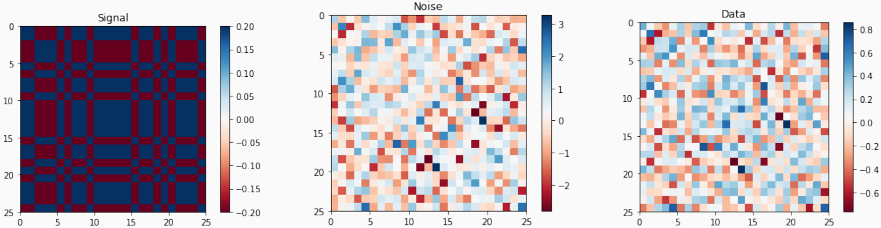
Infer matrix $\boldsymbol{X} \in \mathbb{R}^{N \times M}$ from the noisy matrix $\boldsymbol{Y} \in \mathbb{R}^{N \times N}$

Structure: $\boldsymbol{X} \in \mathbb{R}^{N \times M}$ matrix with i.i.d. entries from a known (centered) prior $\mathbb{P}_X$.

Spiked Wigner Matrix: Johnstone '00

Infer matrix $\boldsymbol{X} \in \mathbb{R}^{N \times M}$ from the noisy matrix $\boldsymbol{Y} \in \mathbb{R}^{N \times N}$

Structure: $\boldsymbol{X} \in \mathbb{R}^{N \times M}$ matrix with i.i.d. entries from a known (centered) prior $\mathbb{P}_X$.

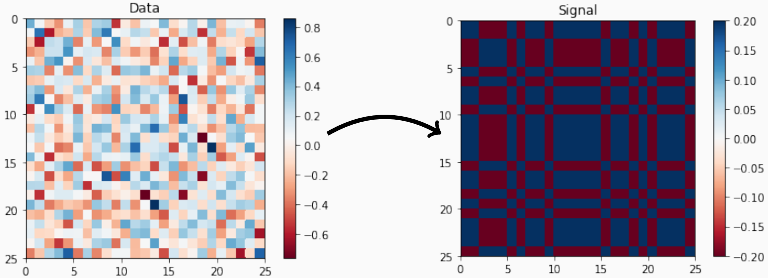


$$\frac{\sqrt{\lambda}}{\sqrt{N}} \boldsymbol{X}\boldsymbol{X}^{\intercal} + \boldsymbol{Z} = \boldsymbol{Y}$$

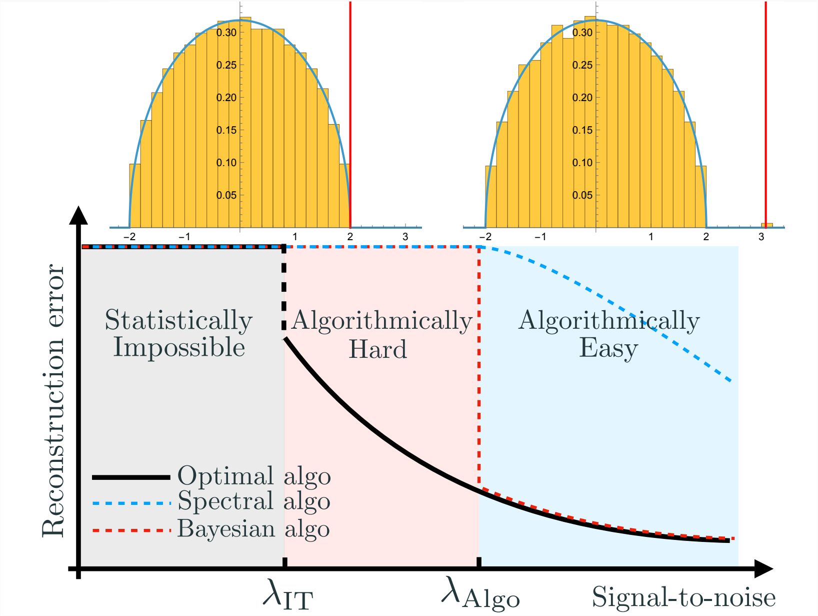
Spiked Wigner Matrix: Johnstone '00

Infer matrix $\boldsymbol{X} \in \mathbb{R}^{N\times M}$ from the noisy matrix $\boldsymbol{Y} \in \mathbb{R}^{N\times N}$

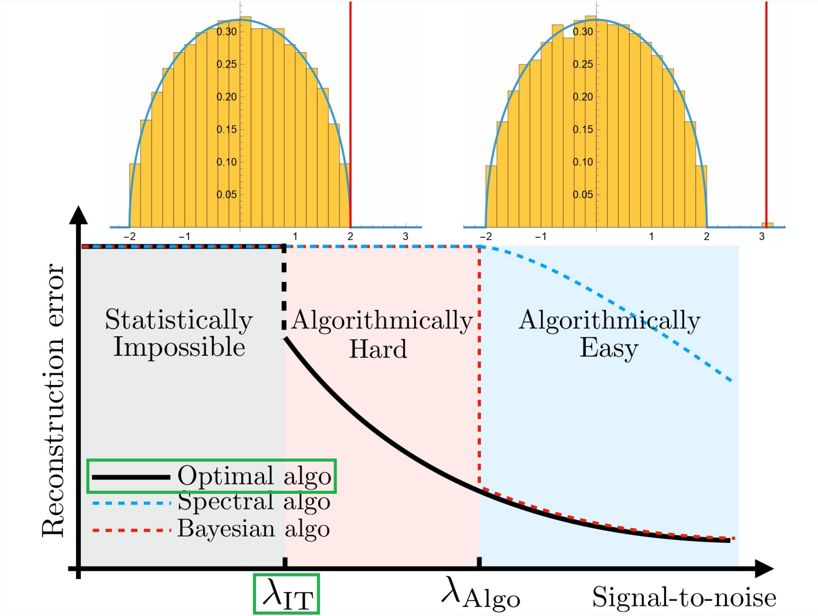
Structure: $\boldsymbol{X} \in \mathbb{R}^{N\times M}$ matrix with i.i.d. entries from a known (centered) prior $\mathbb{P}_X$.



Main Question: Phase Diagram



Main Question: Phase Diagram



Bayesian Inference

Goals: Compute the matrix mean squared error of the matrix $\boldsymbol{X}\boldsymbol{X}^\intercal$

$$
\begin{aligned}
\mathrm{MMSE}_N(\lambda) &= \frac{1}{NM^2} \sum_{i,j=1}^{N} \min_{\theta} \mathbb{E}[(X_i \cdot X_j - \theta_{i,j}(\boldsymbol{Y}))^2] \\
&= \frac{1}{NM^2} \mathbb{E} \| \boldsymbol{X}\boldsymbol{X}^\intercal - \langle \boldsymbol{x}\boldsymbol{x}^\intercal \rangle \|_2^2
\end{aligned}
$$

where $\langle \cdot \rangle$ denotes the average with respect to the posterior.

Bayesian Inference

Goals: Compute the matrix mean squared error of the matrix $\boldsymbol{X}\boldsymbol{X}^\intercal$

$$\mathrm{MMSE}_N(\lambda) = \frac{1}{NM^2}\sum_{i,j=1}^{N} \min_{\theta} \mathbb{E}[(X_i \cdot X_j - \theta_{i,j}(\boldsymbol{Y}))^2]$$
$$= \frac{1}{NM^2}\mathbb{E}\|\boldsymbol{X}\boldsymbol{X}^\intercal - \langle \boldsymbol{x}\boldsymbol{x}^\intercal \rangle\|_2^2$$

where $\langle \cdot \rangle$ denotes the average with respect to the posterior.

Posterior:

$$d\,\mathbb{P}(x \mid Y) = \frac{\exp(-\frac{1}{4}\|\boldsymbol{Y} - \frac{\sqrt{\lambda}}{N}\boldsymbol{x}\boldsymbol{x}^\intercal\|_2^2)}{C(Y)} d\,\mathbb{P}_X^{\otimes N \times M}(\boldsymbol{x}).$$

Statistical Goals

Connection: IMMSE — Relation

$$\frac{d}{d\lambda}\frac{1}{NM}I(\boldsymbol{X};\boldsymbol{Y}) = \frac{1}{4}\mathrm{MMSE}_N(\lambda)$$

Mutual Information: Compute the mutual information

$$I(\boldsymbol{X};\boldsymbol{Y}) := \frac{\lambda\|\mathbb{E}[\boldsymbol{X}^\intercal\boldsymbol{X}]\|_2^2}{4} - \frac{1}{NM}\mathbb{E}_{Z,X}\ln Z_{N,M} \tag{1}$$

where

$$\frac{1}{NM}\mathbb{E}_{Z,X}\ln Z_{N,M}$$

is the *free entropy*.

Statistical Physics

Goals: Compute the free entropy

$$F_{N,M}(\lambda) := \frac{1}{NM} \mathbb{E}_{Z,X} \ln Z_{N,M}$$

where

$$Z_{N,M} := \int_{\mathbb{R}^{N \times M}} \exp(H_N(x)) \, \mathrm{d}\mathbb{P}_X^{\otimes NM}(\boldsymbol{x})$$

denotes the partition function associated with the Hamiltonian

$$\begin{aligned} H_N(\boldsymbol{x}) &:= -\frac{1}{2} \left\| \boldsymbol{Y} - \sqrt{\frac{\lambda}{N}} \boldsymbol{x}\boldsymbol{x}^\intercal \right\|_2^2 + \mathrm{cons} \\ &= \frac{1}{2} \operatorname{tr} \left(\sqrt{\frac{\lambda}{N}} \boldsymbol{Z}\boldsymbol{x}\boldsymbol{x}^\intercal + \frac{\lambda}{N} \boldsymbol{X}\boldsymbol{X}^\intercal \boldsymbol{x}\boldsymbol{x}^\intercal - \frac{\lambda}{2N} \boldsymbol{x}\boldsymbol{x}^\intercal \boldsymbol{x}\boldsymbol{x}^\intercal \right). \end{aligned}$$

The Dimension of the Signal Matrix

We want to study the *free entropy* for different ranks with both rotationally invariant and non-rotationally invariant priors:

- Rank 1 / Finite Rank Case: $M = k < \infty$.

The Dimension of the Signal Matrix

We want to study the *free entropy* for different ranks with both rotationally invariant and non-rotationally invariant priors:

- Rank 1 / Finite Rank Case: $M = k < \infty$.
- Sublinear Rank: $M(N) = o(N)$, $M(N) \to \infty$.

The Dimension of the Signal Matrix

We want to study the *free entropy* for different ranks with both rotationally invariant and non-rotationally invariant priors:

- Rank 1 / Finite Rank Case: $M = k < \infty$.
- Sublinear Rank: $M(N) = o(N)$, $M(N) \to \infty$.
- Extensive Rank: $M(N) = \alpha N$, $\alpha > 0$.

The Dimension of the Signal Matrix

We want to study the *free entropy* for different ranks with both rotationally invariant and non-rotationally invariant priors:

- Rank 1 / Finite Rank Case: $M = k < \infty$.
- Sublinear Rank: $M(N) = o(N)$, $M(N) \to \infty$.
- Extensive Rank: $M(N) = \alpha N$, $\alpha > 0$.

The problem gets harder to analyze as we transition from rank 1 to extensive rank problems

The Dimension of the Signal Matrix

We want to study the *free entropy* for different ranks with both rotationally invariant and non-rotationally invariant priors:

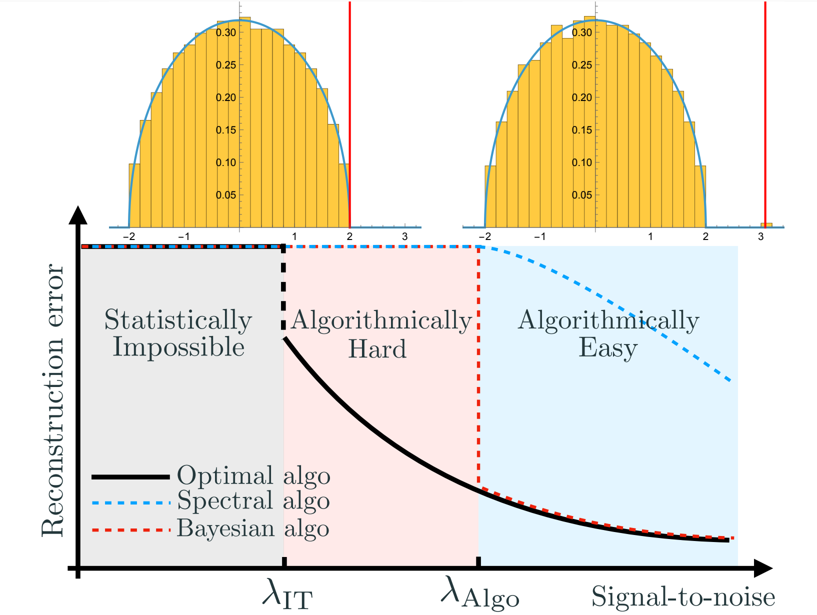
- Rank 1 / Finite Rank Case: $M = k < \infty$.
- Sublinear Rank: $M(N) = o(N)$, $M(N) \to \infty$.
- Extensive Rank: $M(N) = \alpha N$, $\alpha > 0$.

The problem gets harder to analyze as we transition from rank 1 to extensive rank problems

Extensive Rank Setting: Understand extensive-width shallow neural networks (Barbier et al.), Matrix least squares (Ma–Fan)

Table of Contents

Rank 1: Phase Diagram



Known Results: Rank 1 Case

Consider the scenario where $M = 1$.

Theorem 1 (Barbier et al, Lelarge - Miolane, El Alaoui–Krzakala)

$$\lim_{N\to\infty} \frac{1}{N}\mathbb{E}_{Z,X} \ln Z_{N,1}(\lambda) = \sup_q \varphi_1(q).$$

Replica Symmetric Functional:

$$\varphi_1(q) = -\frac{\lambda q^2}{4} + \mathbb{E}\ln\left[\int \exp\left(\sqrt{\lambda q} zx + \lambda q x X - \frac{\lambda q^2 x^2}{2}\right) d\,\mathbb{P}_X(x)\right].$$

Known Results: Rank 1 Case

Consider the scenario where $M = 1$.

Theorem 1 (Barbier et al, Lelarge - Miolane, El Alaoui–Krzakala)

$$\lim_{N\to\infty} \frac{1}{N}\mathbb{E}_{Z,X} \ln Z_{N,1}(\lambda) = \sup_q \varphi_1(q).$$

Replica Symmetric Functional:

$$\varphi_1(q) = -\frac{\lambda q^2}{4} + \mathbb{E} \ln \left[\int \exp\left(\sqrt{\lambda q} z x + \lambda q x X - \frac{\lambda q^2 x^2}{2} \right) d\,\mathbb{P}_X(x) \right].$$

Overlap Concentration (Barbier): Nishimori identity

$$\mathbb{E}\left\langle \left(\frac{\boldsymbol{x} \cdot \boldsymbol{X}}{N} - q \right)^2 \right\rangle \to 0$$

where $\langle \cdot \rangle$ is the average with respect to the posterior.

Known Results: Rank 1 Case

Consider the scenario where $M = 1$.

Theorem 1 (Barbier et al, Lelarge - Miolane, El Alaoui–Krzakala)

$$\lim_{N\to\infty} \frac{1}{N}\mathbb{E}_{Z,X} \ln Z_{N,1}(\lambda) = \sup_q \varphi_1(q).$$

Replica Symmetric Functional:

$$\varphi_1(q) = -\frac{\lambda q^2}{4} + \mathbb{E} \ln \left[\int \exp\left(\sqrt{\lambda q} zx + \lambda q x X - \frac{\lambda q^2 x^2}{2} \right) d\,\mathbb{P}_X(x) \right].$$

Overlap Concentration (Barbier): Nishimori identity

$$\mathbb{E}\left\langle \left(\frac{\boldsymbol{x} \cdot \boldsymbol{X}}{N} - q \right)^2 \right\rangle \to 0$$

where $\langle \cdot \rangle$ is the average with respect to the posterior.

Order Parameter: The maximizing q satisfies

$$q = \lim_{N\to\infty} N^{-1}\mathbb{E}\langle \boldsymbol{x} \cdot \boldsymbol{X} \rangle \in \mathbb{R}$$

Known Results: Rank 1 Case

Consider the scenario where $M = 1$.

Theorem 1 (Barbier et al, Lelarge - Miolane, El Alaoui–Krzakala)

$$\lim_{N\to\infty} \frac{1}{N}\mathbb{E}_{Z,X} \ln Z_{N,1}(\lambda) = \sup_q \varphi_1(q).$$

Replica Symmetric Functional:

$$\varphi_1(q) = -\frac{\lambda q^2}{4} + \mathbb{E} \ln \Big[\int \exp\Big(\sqrt{\lambda q} z x + \lambda q x X - \frac{\lambda q^2 x^2}{2} \Big)\, d\,\mathbb{P}_X(x) \Big].$$

Overlap Concentration (Barbier): Nishimori identity

$$\mathbb{E}\Big\langle \Big(\frac{\boldsymbol{x}\cdot\boldsymbol{X}}{N} - q \Big)^2 \Big\rangle \to 0$$

where $\langle\cdot\rangle$ is the average with respect to the posterior.

Order Parameter: The maximizing q satisfies

$$q = \lim_{N\to\infty} N^{-1}\mathbb{E}\langle \boldsymbol{x}\cdot\boldsymbol{X}\rangle \in \mathbb{R}$$

Phase Diagram: Analyzing the maximizer (Lesieur et al.)

Known Results: Rank k Case

Consider the scenario where $M = k < \infty$.

Theorem 2 (Lelarge - Miolane, Barbier - K - Rahman)

$$\lim_{N\to\infty} \frac{1}{Nk}\mathbb{E}_{Z,X} \ln Z_{N,k}(\lambda) = \sup_{\mathbf{Q}} \frac{1}{k}\varphi_k(\mathbf{Q}) = \sup_{q} \varphi_1(q).$$

Replica Symmetric Functional: Let $\mathbf{Q} \in \mathbb{R}^{k\times k}$

$$\varphi_k(\mathbf{Q}) = -\frac{\lambda \operatorname{tr}(\mathbf{Q}^2)}{4} + \mathbb{E} \ln \left[\int \exp\left(\sqrt{\lambda \mathbf{Q}} \boldsymbol{z} \cdot \boldsymbol{x} + \lambda \boldsymbol{x}^\intercal \mathbf{Q} X - \frac{\lambda \boldsymbol{x}^\intercal \mathbf{Q} \boldsymbol{x}}{2} \right) d\,\mathbb{P}_X^{\otimes k}(x) \right].$$

Known Results: Rank k Case

Consider the scenario where $M = k < \infty$.

Theorem 2 (Lelarge - Miolane, Barbier - K - Rahman)

$$\lim_{N\to\infty} \frac{1}{Nk}\mathbb{E}_{Z,X} \ln Z_{N,k}(\lambda) = \sup_{\mathbf{Q}} \frac{1}{k}\varphi_k(\mathbf{Q}) = \sup_{q} \varphi_1(q).$$

Replica Symmetric Functional: Let $\mathbf{Q} \in \mathbb{R}^{k\times k}$

$$\varphi_k(\mathbf{Q}) = -\frac{\lambda\,\mathrm{tr}(\mathbf{Q}^2)}{4} + \mathbb{E}\ln\left[\int \exp\left(\sqrt{\lambda\mathbf{Q}}\boldsymbol{z}\cdot\boldsymbol{x} + \lambda\boldsymbol{x}^\intercal\mathbf{Q}X - \frac{\lambda\boldsymbol{x}^\intercal\mathbf{Q}\boldsymbol{x}}{2}\right) d\,\mathbb{P}_X^{\otimes k}(x)\right].$$

Order Parameter: The maximizing $\mathbf{Q}$ satisfies

$$\mathbf{Q} = N^{-1}\lim_{N\to\infty}\mathbb{E}\langle \boldsymbol{X}^\intercal\boldsymbol{x}\rangle \in \mathbb{R}^{k\times k}$$

Table of Contents

Known Results: Sublinear Rank Case

Intermediate Problem: Consider the scenario where $M = o(N)$.

1. Configurations $\boldsymbol{x} \in \mathbb{R}^{N \times M}$ now grow simultaneously in both coordinates.

Known Results: Sublinear Rank Case

Intermediate Problem: Consider the scenario where $M = o(N)$.

1. Configurations $\boldsymbol{x} \in \mathbb{R}^{N \times M}$ now grow simultaneously in both coordinates.
2. Concentration stills holds if M grows slowly.

Known Results: Sublinear Rank Case

Intermediate Problem: Consider the scenario where $M = o(N)$.

1. Configurations $\boldsymbol{x} \in \mathbb{R}^{N \times M}$ now grow simultaneously in both coordinates.
2. Concentration stills holds if M grows slowly.
3. The order parameters will be independent of N due to symmetries.

Main Result

Consider the scenario where $M = \log(N)$.

Theorem 3 (Limiting Free Energy (Barbier - K - Rahman))

If $\boldsymbol{X}$ has i.i.d. from a bounded centered distribution and φ is sufficiently regular in λ, then

$$\lim_{N\to\infty} F_{NM}(\lambda) = \sup_q \varphi_1(q).$$

Replica Symmetric Functional:

$$\varphi_1(q) = -\frac{\lambda q^2}{4} + \mathbb{E}\ln\left[\int \exp\left(\sqrt{\lambda q}zx + \lambda qxX - \frac{\lambda q^2 x^2}{2}\right) d\,\mathbb{P}_X(x)\right].$$

Main Result

Consider the scenario where $M = \log(N)$.

Theorem 3 (Limiting Free Energy (Barbier - K - Rahman))

If $\boldsymbol{X}$ *has i.i.d. from a bounded centered distribution and* φ *is sufficiently regular in* λ*, then*

$$\lim_{N\to\infty} F_{NM}(\lambda) = \sup_q \varphi_1(q).$$

Replica Symmetric Functional:

$$\varphi_1(q) = -\frac{\lambda q^2}{4} + \mathbb{E}\ln\left[\int \exp\left(\sqrt{\lambda q} zx + \lambda q x X - \frac{\lambda q^2 x^2}{2}\right) d\,\mathbb{P}_X(x)\right].$$

Spherical priors / Rotational Invariant Priors: Husson-Ko for $M = o(N)$

Table of Contents

Extensive Rank Case

Consider the scenario where $M = \alpha N$.

1. Configurations $\boldsymbol{x} \in \mathbb{R}^{N \times M}$ now grow simultaneously in both coordinates.

Extensive Rank Case

Consider the scenario where $M = \alpha N$.

1. Configurations $\boldsymbol{x} \in \mathbb{R}^{N \times M}$ now grow simultaneously in both coordinates.
2. Concentration fails because M grows too fast.

Extensive Rank Case

Consider the scenario where $M = \alpha N$.

1. Configurations $\boldsymbol{x} \in \mathbb{R}^{N \times M}$ now grow simultaneously in both coordinates.
2. Concentration fails because M grows too fast.
3. There is no good notion of the order parameters $Q \in \mathbb{R}^{M \times M}$ in the limit.

Known Results: Rotational Invariant Prior

Gaussian Prior (Unstructured): $X_{ij} \sim N(0, 1)$

Known Results: Rotational Invariant Prior

Gaussian Prior (Unstructured): $X_{ij} \sim N(0, 1)$

MMSE: Using HCIZ integrals (Matytsin, Guionnet–Zeitouni),

$$\lim_{N \to \infty} \mathrm{MMSE}_N(\lambda) = \frac{1}{\lambda\alpha}\left(1 - \frac{4\pi^2}{3}\int \rho_Y(y)^3 \, dy\right)$$

where ρ_Y is the limiting spectral distribution of $\boldsymbol{Y}$.

Known Results: Rotational Invariant Prior

Gaussian Prior (Unstructured): $X_{ij} \sim N(0, 1)$

MMSE: Using HCIZ integrals (Matytsin, Guionnet–Zeitouni),

$$\lim_{N \to \infty} \mathrm{MMSE}_N(\lambda) = \frac{1}{\lambda\alpha}\left(1 - \frac{4\pi^2}{3} \int \rho_Y(y)^3 \, dy\right)$$

where ρ_Y is the limiting spectral distribution of $\boldsymbol{Y}$.

Mutual Information: Using HCIZ integrals,

$$\lim_{N \to \infty} \frac{1}{MN} I(\boldsymbol{XX}; \boldsymbol{Y}) = \frac{1}{8\alpha} + \frac{1}{2\alpha} \int \rho_Y(x)\rho_Y(y) \ln|x - y| \, dxdy.$$

Known Results: Rotational Invariant Prior

Gaussian Prior (Unstructured): $X_{ij} \sim N(0,1)$

MMSE: Using HCIZ integrals (Matytsin, Guionnet–Zeitouni),

$$\lim_{N\to\infty} \mathrm{MMSE}_N(\lambda) = \frac{1}{\lambda\alpha}\left(1 - \frac{4\pi^2}{3}\int \rho_Y(y)^3\,dy\right)$$

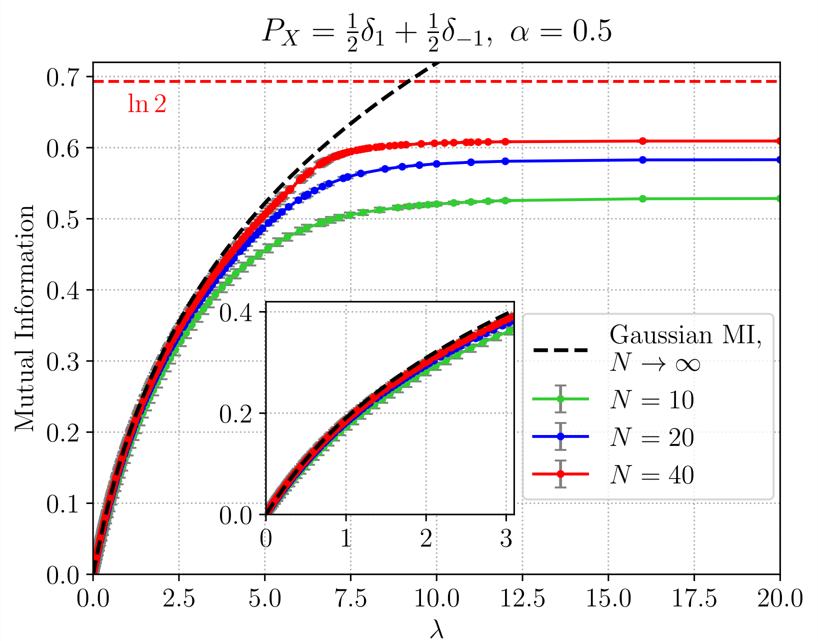
where ρ_Y is the limiting spectral distribution of $\boldsymbol{Y}$.

Mutual Information: Using HCIZ integrals,

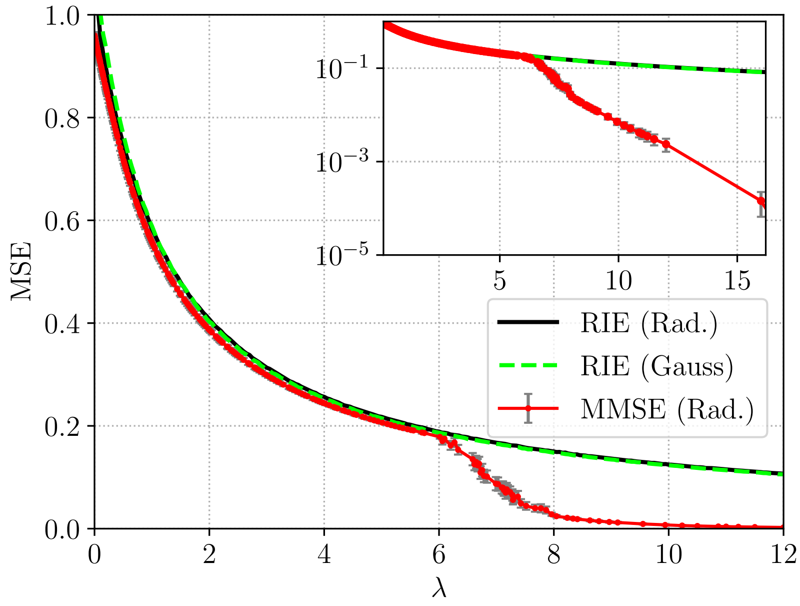
$$\lim_{N\to\infty} \frac{1}{MN} I(\boldsymbol{XX};\boldsymbol{Y}) = \frac{1}{8\alpha} + \frac{1}{2\alpha}\int \rho_Y(x)\rho_Y(y)\ln|x-y|\,dxdy.$$

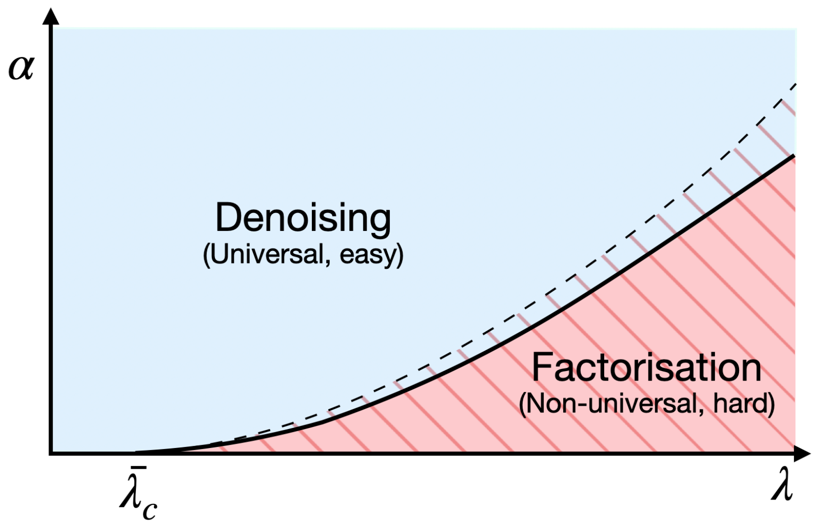
Conjecture: This is correct for all priors (e.g. $\mathbb{P}_X$ is Rademacher) (Semerjian).

Numerical Results: Non-Rotational Invariant Prior



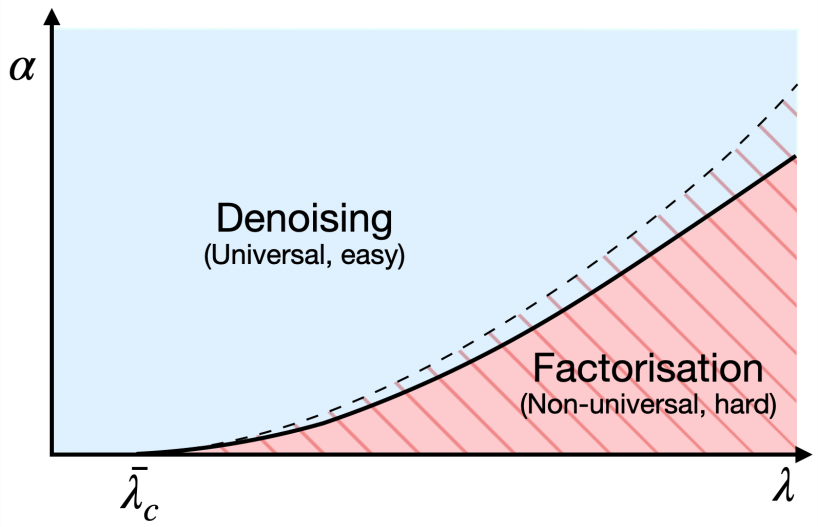
Numerical Results: Non-Rotational Invariant Prior





α
Denoising
(Universal, easy)
Factorisation
(Non-universal, hard)
$\bar{\lambda}_c$
λ

Qualitative Phase Diagram



Random Matrix Theory $\longrightarrow$ Spin Glass Theory

Results: Prediction of the MI

Result 1 (Barbier - Camilli - K - Okajima)

If the prior is factorized there exists a $\lambda_c > 0$ *such that for all* $\lambda > \lambda_c$,

$$\lim_{N\to\infty} I(\boldsymbol{X};\boldsymbol{Y}) = \mathrm{extr}_{-}\{\iota(r,q;\alpha,\lambda)\}$$

where

$$\iota(r,q;\alpha,\lambda) = \frac{rq}{2} + \frac{1}{4\alpha}\ln(1+\lambda\alpha(1-q^2)) \\ - \mathbb{E}\ln\left[\int \exp\left(\sqrt{\lambda r}zx + \lambda rxX - \frac{\lambda r^2x^2}{2}\right) d\,\mathbb{P}_X(x)\right]$$

Results: Prediction of the MI

Result 2 (Barbier - Camilli - K - Okajima)

If the prior is factorized there exists a $\lambda_c > 0$ *such that for all* $\lambda > \lambda_c$,

$$\lim_{N\to\infty} I(\boldsymbol{X};\boldsymbol{Y}) = \mathrm{extr}_{-}\{\iota(r,q;\alpha,\lambda)\}$$

where

$$\iota(r,q;\alpha,\lambda) = \frac{rq}{2} + \frac{1}{4\alpha}\ln(1+\lambda\alpha(1-q^2)) - \mathbb{E}\ln\left[\int \exp\left(\sqrt{\lambda r}zx + \lambda r x X - \frac{\lambda r^2 x^2}{2}\right) d\,\mathbb{P}_X(x)\right]$$

Remark: Same result if you use the replica trick assuming a Gaussian ansatz in Sakata–Kabashima.

Results: Prediction of the Denoising–Factorization Transition

Rotationally Invariant Mutual Information:

$$\iota_D(\lambda) = \frac{1}{8\alpha} + \frac{1}{2\alpha} \int \rho_Y(x)\rho_Y(y) \ln |x - y| \, dxdy$$

Non-Rotationally Invariant Mutual Information:

$$\iota_F(\lambda) = \mathrm{extr}_{-}\{\iota(r, q; \alpha, \lambda)\}.$$

Results: Prediction of the Denoising–Factorization Transition

Rotationally Invariant Mutual Information:

$$\iota_D(\lambda) = \frac{1}{8\alpha} + \frac{1}{2\alpha} \int \rho_Y(x)\rho_Y(y) \ln|x - y|\, dxdy$$

Non-Rotationally Invariant Mutual Information:

$$\iota_F(\lambda) = \mathrm{extr}_{-}\{\iota(r, q; \alpha, \lambda)\}.$$

Result 3 (Barbier - Camilli - K - Okajima)

If the prior is factorized then for all $\lambda \geq 0$,

$$\lim_{N\to\infty} I(\boldsymbol{X}; \boldsymbol{Y}) = \min(\iota_D, \iota_F).$$

Results: Prediction of the Denoising–Factorization Transition

Rotationally Invariant Mutual Information:

$$\iota_D(\lambda) = \frac{1}{8\alpha} + \frac{1}{2\alpha} \int \rho_Y(x)\rho_Y(y) \ln|x-y| \, dxdy$$

Non-Rotationally Invariant Mutual Information:

$$\iota_F(\lambda) = \mathrm{extr}_{-}\{\iota(r, q; \alpha, \lambda)\}.$$

Result 4 (Barbier - Camilli - K - Okajima)

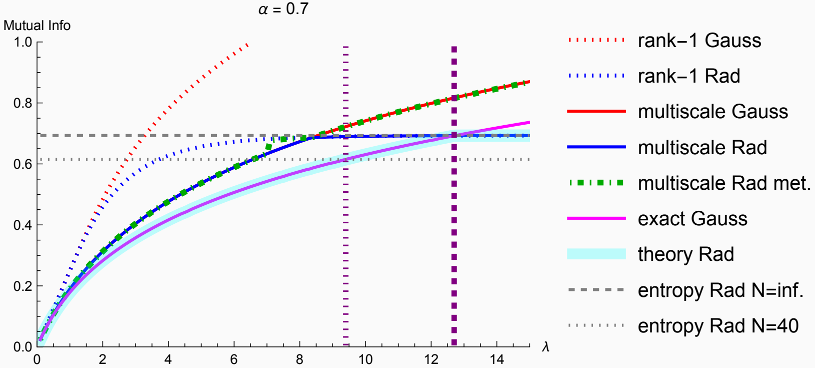
If the prior is factorized then for all $\lambda \geq 0$,

$$\lim_{N\to\infty} I(\boldsymbol{X}; \boldsymbol{Y}) = \min(\iota_D, \iota_F).$$

Critical λ:

$$\lambda_c = \sup\{\lambda \geq 0 \mid \iota_D \leq \iota_F\}.$$

Results: Comparison with Simulations



Results: Comparison with Simulations

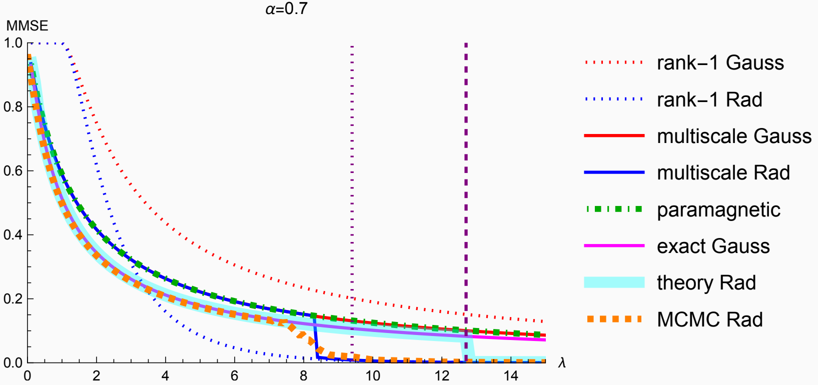


Table of Contents

Challenges: Extensive Rank Case

Goal: Compute the theoretical curve for the MI when λ is large.

Challenges: Extensive Rank Case

Goal: Compute the theoretical curve for the MI when λ is large.

Strategy: Adapt the techniques from spin glasses (Guerra, Aizenman–Sims–Starr, Talagrand, Panchenko, Lelarge–Miolane)

Review: Rank k Case

Lower Bound - Guerra's Interpolation: Gaussian interpolation with the replica symmetric functional.

Review: Rank k Case

Cavity Method: Reduce the problem from N dimensions to k dimensions.

Review: Rank k Case

Cavity Method: Reduce the problem from N dimensions to k dimensions.

Write

$$\limsup \frac{1}{Nk}\mathbb{E}_{Z,X} \ln Z_{N,k}(\lambda) \leq \limsup \frac{1}{k}\Big(\mathbb{E}_{Z,X} \ln Z_{N+1,k}(\lambda) - \mathbb{E}_{Z,X} \ln Z_{N,k}(\lambda)\Big)$$

Review: Rank k Case

Cavity Method: Reduce the problem from N dimensions to k dimensions.

Write

$$\limsup \frac{1}{Nk}\mathbb{E}_{Z,X} \ln Z_{N,k}(\lambda) \leq \limsup \frac{1}{k}\Big(\mathbb{E}_{Z,X} \ln Z_{N+1,k}(\lambda) - \mathbb{E}_{Z,X} \ln Z_{N,k}(\lambda)\Big)$$

Split into cavity fields $(\mathbf{x}, \boldsymbol{w}) \in \mathbb{R}^{N\times k} \times \mathbb{R}^k$

$$H_{N+1}(\mathbf{x}) = z_N(\mathbf{x}, \boldsymbol{w}) + H'_N(\mathbf{x}), \qquad H_N(\mathbf{x}) = y_N(\mathbf{x}) + H'_N(\mathbf{x}).$$

Review: Rank k Case

Cavity Method: Reduce the problem from N dimensions to k dimensions.

Write

$$\limsup \frac{1}{Nk}\mathbb{E}_{Z,X} \ln Z_{N,k}(\lambda) \leq \limsup \frac{1}{k}\Big(\mathbb{E}_{Z,X} \ln Z_{N+1,k}(\lambda) - \mathbb{E}_{Z,X} \ln Z_{N,k}(\lambda)\Big)$$

Split into cavity fields $(\mathbf{x}, \boldsymbol{w}) \in \mathbb{R}^{N\times k} \times \mathbb{R}^k$

$$H_{N+1}(\mathbf{x}) = z_N(\mathbf{x}, \boldsymbol{w}) + H'_N(\mathbf{x}), \qquad H_N(\mathbf{x}) = y_N(\mathbf{x}) + H'_N(\mathbf{x}).$$

$$\mathbb{E}_{Z,X} \ln Z_{N+1,k}(\lambda) - \mathbb{E}_{Z,X} \ln Z_{N,k}(\lambda)$$
$$= \mathbb{E} \ln \int e^{z_N(\mathbf{x},w)+H'_N(\mathbf{x})} d\,\mathbb{P}_X^{\otimes k}(w) d\,\mathbb{P}_X^{\otimes Nk}(x) - \mathbb{E} \ln \int e^{y_N(\mathbf{x})+H'_N(\mathbf{x})}\, d\,\mathbb{P}_X^{\otimes Nk}(x)$$

Review: Rank k Case

Cavity Method: Reduce the problem from N dimensions to k dimensions.

Write

$$\limsup \frac{1}{Nk}\mathbb{E}_{Z,X} \ln Z_{N,k}(\lambda) \leq \limsup \frac{1}{k}\Big(\mathbb{E}_{Z,X} \ln Z_{N+1,k}(\lambda) - \mathbb{E}_{Z,X} \ln Z_{N,k}(\lambda)\Big)$$

Split into cavity fields $(\mathbf{x}, \boldsymbol{w}) \in \mathbb{R}^{N\times k} \times \mathbb{R}^k$

$$H_{N+1}(\mathbf{x}) = z_N(\mathbf{x}, \boldsymbol{w}) + H_N'(\mathbf{x}), \qquad H_N(\mathbf{x}) = y_N(\mathbf{x}) + H_N'(\mathbf{x}).$$

$$\mathbb{E}_{Z,X} \ln Z_{N+1,k}(\lambda) - \mathbb{E}_{Z,X} \ln Z_{N,k}(\lambda)$$

$$= \mathbb{E} \ln \int e^{z_N(\mathbf{x},w)+H_N'(\mathbf{x})} d\,\mathbb{P}_X^{\otimes k}(w) d\,\mathbb{P}_X^{\otimes Nk}(x) - \mathbb{E} \ln \int e^{y_N(\mathbf{x})+H_N'(\mathbf{x})}\, d\,\mathbb{P}_X^{\otimes Nk}(x)$$

$$= \mathbb{E} \ln \left\langle \int e^{z_N(\mathbf{x},w)} d\,\mathbb{P}_X^{\otimes k}(w) \right\rangle' - \mathbb{E} \ln \left\langle e^{y_N(\mathbf{x})} \right\rangle'$$

Review: Rank k Case

Cavity Method: Reduce the problem from N dimensions to k dimensions.

Write

$$\limsup \frac{1}{Nk}\mathbb{E}_{Z,X} \ln Z_{N,k}(\lambda) \leq \limsup \frac{1}{k}\Big(\mathbb{E}_{Z,X} \ln Z_{N+1,k}(\lambda) - \mathbb{E}_{Z,X} \ln Z_{N,k}(\lambda)\Big)$$

Split into cavity fields $(\mathbf{x}, \boldsymbol{w}) \in \mathbb{R}^{N\times k} \times \mathbb{R}^k$

$$H_{N+1}(\mathbf{x}) = z_N(\mathbf{x}, \boldsymbol{w}) + H'_N(\mathbf{x}), \qquad H_N(\mathbf{x}) = y_N(\mathbf{x}) + H'_N(\mathbf{x}).$$

Concentration of $X^\intercal x$ under $\langle\cdot\rangle'$

$$\mathbb{E}_{Z,X} \ln Z_{N+1,k}(\lambda) - \mathbb{E}_{Z,X} \ln Z_{N,k}(\lambda)$$

$$= \mathbb{E}\ln \int e^{z_N(\mathbf{x},w)+H'_N(\mathbf{x})} d\,\mathbb{P}_X^{\otimes k}(w) d\,\mathbb{P}_X^{\otimes Nk}(x) - \mathbb{E}\ln \int e^{y_N(\mathbf{x})+H'_N(\mathbf{x})}\, d\,\mathbb{P}_X^{\otimes Nk}(x)$$

$$\approx \mathbb{E}\ln\left[\int \exp\left(\sqrt{\lambda \mathbf{Q}}\boldsymbol{z}\cdot\boldsymbol{w} + \lambda \boldsymbol{w}^\intercal \mathbf{Q}\boldsymbol{x} - \frac{\lambda \boldsymbol{w}^\intercal \mathbf{Q}\boldsymbol{w}}{2}\right) d\,\mathbb{P}_X^{\otimes k}(w)\right] - \frac{\lambda\,\mathrm{tr}(\mathbf{Q}^2)}{4}$$

Review: Sublinear Rank Case

Consider the case where $M = o(N)$.

Challenges:

- Deal with increments in N and M simultaneously

Review: Sublinear Rank Case

Consider the case where $M = o(N)$.

Challenges:

- Deal with increments in N and M simultaneously
- Overlap concentration still holds if M grows sufficiently slowly

Technique 1: Multiscale Cavity Method

Theorem 4 (Barbier – K – Rahman)

For scaling parameters $\alpha > 0$, $\gamma \geq 0$, *and* $M_N = \lfloor \alpha N^\gamma \rfloor$, *we have*

$$\limsup_{N\to\infty} F_N(\lambda) \leq \frac{1}{1+\gamma} \limsup_{N\to\infty} \frac{\Delta_{\mathrm{row}}}{M} + \frac{\gamma}{1+\gamma} \limsup_{N\to\infty} \frac{\Delta_{\mathrm{col}}}{N} \tag{2}$$

and

$$\liminf_{N\to\infty} F_N(\lambda) \geq \frac{1}{1+\gamma} \liminf_{N\to\infty} \frac{\Delta_{\mathrm{row}}}{M} + \frac{\gamma}{1+\gamma} \liminf_{N\to\infty} \frac{\Delta_{\mathrm{col}}}{N}, \tag{3}$$

where

$$\Delta_{\mathrm{row}}(N) := \mathbb{E}_{Z,X} \ln Z_{N+1,M_{N+1}} - \mathbb{E}_{Z,X} \ln Z_{N,M_{N+1}}, \tag{4}$$

$$\Delta_{\mathrm{col}}(N) := \mathbb{E}_{Z,X} \ln Z_{N,M_N+1} - \mathbb{E}_{Z,X} \ln Z_{N,M_N}. \tag{5}$$

Technique 1: Multiscale Cavity Method

Theorem 4 (Barbier – K – Rahman)

For scaling parameters $\alpha > 0$, $\gamma \geq 0$, *and* $M_N = \lfloor \alpha N^\gamma \rfloor$, *we have*

$$\limsup_{N\to\infty} F_N(\lambda) \leq \frac{1}{1+\gamma} \limsup_{N\to\infty} \frac{\Delta_{\mathrm{row}}}{M} + \frac{\gamma}{1+\gamma} \limsup_{N\to\infty} \frac{\Delta_{\mathrm{col}}}{N} \tag{2}$$

and

$$\liminf_{N\to\infty} F_N(\lambda) \geq \frac{1}{1+\gamma} \liminf_{N\to\infty} \frac{\Delta_{\mathrm{row}}}{M} + \frac{\gamma}{1+\gamma} \liminf_{N\to\infty} \frac{\Delta_{\mathrm{col}}}{N}, \tag{3}$$

where

$$\Delta_{\mathrm{row}}(N) := \mathbb{E}_{Z,X} \ln Z_{N+1,M_{N+1}} - \mathbb{E}_{Z,X} \ln Z_{N,M_{N+1}}, \tag{4}$$

$$\Delta_{\mathrm{col}}(N) := \mathbb{E}_{Z,X} \ln Z_{N,M_N+1} - \mathbb{E}_{Z,X} \ln Z_{N,M_N}. \tag{5}$$

Discovered jointly with: Barbier, Rahman, Camilli, Okajima

Review: Sublinear Rank Case

Row Cavity Method: Reduce the problem from NM dimensions to $M = o(N)$ dimensions.

Review: Sublinear Rank Case

Row Cavity Method: Reduce the problem from NM dimensions to $M = o(N)$ dimensions.

Split into cavity fields $(\mathbf{x}, \boldsymbol{w}) \in \mathbb{R}^{N\times M} \times \mathbb{R}^M$

$$H_{N+1}(\mathbf{x}) = z_N(\mathbf{x}, \boldsymbol{w}) + H'_N(\mathbf{x}), \qquad H_N(\mathbf{x}) = y_N(\mathbf{x}) + H'_N(\mathbf{x}).$$

Review: Sublinear Rank Case

Row Cavity Method: Reduce the problem from NM dimensions to $M = o(N)$ dimensions.

Split into cavity fields $(\mathbf{x}, \boldsymbol{w}) \in \mathbb{R}^{N\times M} \times \mathbb{R}^M$

$$H_{N+1}(\mathbf{x}) = z_N(\mathbf{x}, \boldsymbol{w}) + H'_N(\mathbf{x}), \qquad H_N(\mathbf{x}) = y_N(\mathbf{x}) + H'_N(\mathbf{x}).$$

Concentration of $X^\intercal x$ under $\langle\cdot\rangle'$

$$\begin{aligned}
&\frac{1}{M}(\mathbb{E}_{Z,X} \ln Z_{N+1,M}(\lambda) - \mathbb{E}_{Z,M} \ln Z_{N,M}(\lambda)) \\
&= \frac{1}{M}\left(\mathbb{E}\ln\left\langle \int e^{z_N(\mathbf{x},w)}\, d\,\mathbb{P}_X^{\otimes M}(w)\right\rangle' - \mathbb{E}\ln\left\langle e^{y_N(\mathbf{x})}\right\rangle'\right) \\
&\approx \frac{1}{M}\left(\mathbb{E}\ln\left[\int \exp\left(\sqrt{\lambda\mathbf{Q}}\boldsymbol{z}\cdot\boldsymbol{w} + \lambda\boldsymbol{w}^\intercal\mathbf{Q}\boldsymbol{x} - \frac{\lambda\boldsymbol{w}^\intercal\mathbf{Q}\boldsymbol{w}}{2}\right) d\,\mathbb{P}_X^{\otimes M}(w)\right] \right.\\
&\qquad\qquad \left. - \frac{\lambda\,\mathrm{tr}(\mathbf{Q}^2)}{4}\right)
\end{aligned}$$

Review: Sublinear Rank Case

Row Cavity Method: Reduce the problem from NM dimensions to $M = 1$ dimensions.

Split into cavity fields $(\mathbf{x}, \boldsymbol{w}) \in \mathbb{R}^{N\times M} \times \mathbb{R}^{M}$

$$H_{N+1}(\mathbf{x}) = z_N(\mathbf{x}, \boldsymbol{w}) + H'_N(\mathbf{x}), \qquad H_N(\mathbf{x}) = y_N(\mathbf{x}) + H'_N(\mathbf{x}).$$

Concentration of $X^\intercal x$ under $\langle\cdot\rangle'$

$$\frac{1}{M}(\mathbb{E}_{Z,X} \ln Z_{N+1,M}(\lambda) - \mathbb{E}_{Z,M} \ln Z_{N,M}(\lambda))$$

$$= \frac{1}{M}\left(\mathbb{E} \ln \left\langle \int e^{z_N(\mathbf{x},w)} d\,\mathbb{P}_X^{\otimes M}(w)\right\rangle' - \mathbb{E} \ln \left\langle e^{y_N(\mathbf{x})}\right\rangle'\right)$$

$$\approx \frac{1}{M}\left(\mathbb{E} \ln \left[\int \exp\left(\sqrt{\lambda \mathbf{Q}}\boldsymbol{z} \cdot \boldsymbol{w} + \lambda \boldsymbol{w}^\intercal \mathbf{Q}\boldsymbol{x} - \frac{\lambda \boldsymbol{w}^\intercal \mathbf{Q} \boldsymbol{w}}{2}\right) d\,\mathbb{P}_X^{\otimes M}(w)\right] - \frac{\lambda \operatorname{tr}(\mathbf{Q}^2)}{4}\right)$$

$$= \sup_q \varphi_1(q)$$

Technique 2: Rank Dependent Concentration

Theorem 5 (Barbier – K – Rahman)

Let $\mathbb{P}_X$ *be a centered distribution with bounded fourth moment and* s_N *a constant going to* 0*. There exists a perturbation of the posterior and a finite positive constant* C *independent of* M, N *and depending only on properties of* $\mathbb{P}_X$ *such that*

$$\frac{1}{s_N}\int_{s_N}^{2s_N} \mathbb{E}\langle \|\boldsymbol{X}^\intercal x - \langle \boldsymbol{X}^\intercal x\rangle_{N,\epsilon}\|_{\mathrm{F}}^2\rangle_{N,\epsilon}\, \mathrm{d}\epsilon \leq \Gamma(N, M) := \frac{CM^2}{\sqrt{Ns_N}},$$

where the expectation $\mathbb{E}[\,\cdot\,]$ *is taken over* $\widetilde{Z}, X$ *and the randomness in* $\mathbb{P}_{\mathrm{out}}(\,\cdot \mid \boldsymbol{XX}^\intercal)$ *and* $\langle\cdot\rangle$ *is the average with respect to a perturbed posterior.*

Technique 3: Rank 1 Equivalence

Theorem 6 (Barbier - K - Rahman)

Under some regularity hypothesis on the replica symmetric formula. For all SNR $\lambda \geq 0$

$$\sup_Q \frac{1}{k}\varphi_k(\mathbf{Q}) = \sup_q \varphi_1(q). \tag{6}$$

Remember that

$$\varphi_1(q) = -\frac{\lambda q^2}{4} + \mathbb{E}\ln\left[\int \exp\left(\sqrt{\lambda q}zx + \lambda qxX - \frac{\lambda q^2 x^2}{2}\right) d\,\mathbb{P}_X(x)\right].$$

$$\varphi_k(\mathbf{Q}) = -\frac{\lambda\,\mathrm{tr}(\mathbf{Q}^2)}{4} + \mathbb{E}\ln\left[\int \exp\left(\sqrt{\lambda\mathbf{Q}}\boldsymbol{z}\cdot\boldsymbol{x} + \lambda\boldsymbol{x}^\intercal\mathbf{Q}X - \frac{\lambda\boldsymbol{x}^\intercal\mathbf{Q}\boldsymbol{x}}{2}\right) d\,\mathbb{P}_X^{\otimes k}(x)\right].$$

Column Cavity

Need to compute

$$\lim_{N\to\infty} \frac{1}{N}(\mathbb{E}_{Z,X} \ln Z_{N,M_N+1} - \mathbb{E}_{Z,X} \ln Z_{N,M_N})$$

Column Cavity

Need to compute

$$\lim_{N\to\infty} \frac{1}{N}(\mathbb{E}_{Z,X} \ln Z_{N,M_N+1} - \mathbb{E}_{Z,X} \ln Z_{N,M_N})$$

If N is sufficiently large and M grows sufficiently slowly, then

$$\begin{aligned}\frac{1}{N}(\mathbb{E}_{Z,X} \ln Z_{N,M_N+1} - \mathbb{E}_{Z,X} \ln Z_{N,M_N}) &\approx (M_N+1)\sup_q \varphi_1(q) - M_N \sup_q \varphi_1(q) \\ &= \sup_q \varphi_1(q)\end{aligned}$$

Column Cavity

Need to compute

$$\lim_{N\to\infty} \frac{1}{N}(\mathbb{E}_{Z,X} \ln Z_{N,M_N+1} - \mathbb{E}_{Z,X} \ln Z_{N,M_N})$$

If N is sufficiently large and M grows sufficiently slowly, then

$$\begin{aligned}\frac{1}{N}(\mathbb{E}_{Z,X} \ln Z_{N,M_N+1} - \mathbb{E}_{Z,X} \ln Z_{N,M_N}) &\approx (M_N + 1) \sup_q \varphi_1(q) - M_N \sup_q \varphi_1(q) \\ &= \sup_q \varphi_1(q)\end{aligned}$$

Careful proof: We can use the cavity computations again to reduce it to the finite rank cases

Predictions: Extensive Rank Case

Consider the case where $M = \alpha N$.

Challenges:

- Deal with increments in N and M simultaneously

Predictions: Extensive Rank Case

Consider the case where $M = \alpha N$.

Challenges:

- Deal with increments in N and M simultaneously
- Overlap concentration might not hold

Idea: Multiscale–Mean–Field Theory (1st Scale)

Reduction: Reduce the problem from NM dimensions to $M = \alpha N$ dimensions.

Idea: Multiscale–Mean–Field Theory (1st Scale)

Reduction: Reduce the problem from NM dimensions to $M = \alpha N$ dimensions.

Row Cavity: By the Aizenman–Sims–Starr scheme, suffices to compute

$$\lim_{N\to\infty} \frac{1}{M}\mathbb{E}_{\tilde{Y}(\lambda),Y'(\zeta)} \ln \left\langle \int_{\mathbb{R}^M} \exp(H^{row}_{N,M}(\boldsymbol{\eta};\bar{\boldsymbol{X}},\lambda))\, d\,\mathbb{P}_X^{\otimes M}(\boldsymbol{\eta}) \right\rangle'$$

where $H^{row}_{N,M}(\boldsymbol{\eta};\bar{\boldsymbol{X}},\lambda)$ is the loglikelihood of the RLE problem.

Idea: Multiscale–Mean–Field Theory (1st Scale)

Reduction: Reduce the problem from NM dimensions to $M = \alpha N$ dimensions.

Row Cavity: By the Aizenman–Sims–Starr scheme, suffices to compute

$$\lim_{N\to\infty} \frac{1}{M} \mathbb{E}_{\tilde{Y}(\lambda), Y'(\zeta)} \ln \left\langle \int_{\mathbb{R}^M} \exp(H^{row}_{N,M}(\boldsymbol{\eta}; \bar{\boldsymbol{X}}, \lambda))\, d\, \mathbb{P}_X^{\otimes M}(\boldsymbol{\eta}) \right\rangle'$$

where $H^{row}_{N,M}(\boldsymbol{\eta}; \bar{\boldsymbol{X}}, \lambda)$ is the loglikelihood of the RLE problem.

Random Linear Estimation: Cavity fields can be written as a RLE problem

$$\tilde{\boldsymbol{Y}}(\lambda) = \sqrt{\frac{\lambda}{N}} \bar{\boldsymbol{X}} \boldsymbol{H} + \tilde{\boldsymbol{Z}} \tag{7}$$

$$Y'_{ij}(\zeta) = \sqrt{\frac{\zeta}{N}} \boldsymbol{X}_i^\intercal \boldsymbol{X}_j + Z_{ij}, \quad 1 \leq i < j \leq N-1. \tag{8}$$

Goal: Infer the cavity vector $\boldsymbol{H} \in \mathbb{R}^M$ given the side information on the bulk matrix $\bar{\boldsymbol{X}} \in R^{N-1\times M}$.

Idea: Multiscale–Mean–Field Theory (2nd Scale)

Reduction: Reduce the problem from N dimensions to 1 dimensions.

Idea: Multiscale–Mean–Field Theory (2nd Scale)

Reduction: Reduce the problem from N dimensions to 1 dimensions.

Ansatz: Assume the bulk measure is a sample from the effective field

$$\boldsymbol{Y}'_{eff} = \sqrt{\sigma}\boldsymbol{X} + Z'_{eff} \in \mathbb{R}^{(N-1)\times M}.$$

$$\begin{aligned}
&\lim_{N\to\infty} \frac{1}{M}\mathbb{E}_{\tilde{Y}(\lambda),Y'(\zeta)} \ln \left\langle \int_{\mathbb{R}^M} \exp(H^{row}_{N,M}(\boldsymbol{\eta}; \bar{\boldsymbol{X}}, \lambda))\, d\,\mathbb{P}_X^{\otimes M}(\boldsymbol{\eta}) \right\rangle' \\
&\stackrel{?}{=} \lim_{N\to\infty} \frac{1}{M}\mathbb{E}_{\tilde{Y}(\lambda),Y'(\zeta)} \ln \left\langle \int_{\mathbb{R}^M} \exp(H^{row}_{N,M}(\boldsymbol{\eta}; \bar{\boldsymbol{X}}, \lambda))\, d\,\mathbb{P}_X^{\otimes M}(\boldsymbol{\eta}) \right\rangle_{eff}.
\end{aligned}$$

Idea: Multiscale–Mean–Field Theory (2nd Scale)

Reduction: Reduce the problem from N dimensions to 1 dimensions.

Ansatz: Assume the bulk measure is a sample from the effective field

$$\boldsymbol{Y}'_{eff} = \sqrt{\sigma}\boldsymbol{X} + Z'_{eff} \in \mathbb{R}^{(N-1)\times M}.$$

$$\lim_{N\to\infty} \frac{1}{M}\mathbb{E}_{\tilde{Y}(\lambda),Y'(\zeta)} \ln \left\langle \int_{\mathbb{R}^M} \exp(H^{row}_{N,M}(\boldsymbol{\eta};\bar{\boldsymbol{X}},\lambda))\, d\,\mathbb{P}_X^{\otimes M}(\boldsymbol{\eta}) \right\rangle'$$
$$\stackrel{?}{=} \lim_{N\to\infty} \frac{1}{M}\mathbb{E}_{\tilde{Y}(\lambda),Y'(\zeta)} \ln \left\langle \int_{\mathbb{R}^M} \exp(H^{row}_{N,M}(\boldsymbol{\eta};\bar{\boldsymbol{X}},\lambda))\, d\,\mathbb{P}_X^{\otimes M}(\boldsymbol{\eta}) \right\rangle_{eff}.$$

Compute and Optimize: This choice makes the row cavity computable, and we pick σ to satisfy some self consistency criteria to conclude that

$$\lim_{N\to\infty} \frac{1}{M}\mathbb{E}_{\tilde{Y}(\lambda),Y'(\zeta)} \ln \left\langle \int_{\mathbb{R}^M} \exp(H^{row}_{N,M}(\boldsymbol{\eta};\bar{\boldsymbol{X}},\lambda))\, d\,\mathbb{P}_X(\boldsymbol{\eta}) \right\rangle_{eff}$$
$$= \mathrm{extr}_{-}\{\iota(r,q;\alpha,\lambda)\}$$

Towards a Rigorous Result: Behavior of the Overlaps

Key Assumption: The bulk measure is equivalent to the posterior sample from the model

$$\boldsymbol{Y}'_{eff} = \sqrt{\sigma}\boldsymbol{X} + Z'_{eff} \in \mathbb{R}^{(N-1)\times M}.$$

- The ansatz gives an exact result for finite rank

Towards a Rigorous Result: Behavior of the Overlaps

Key Assumption: The bulk measure is equivalent to the posterior sample from the model

$$\boldsymbol{Y}'_{eff} = \sqrt{\sigma}\boldsymbol{X} + Z'_{eff} \in \mathbb{R}^{(N-1)\times M}.$$

- The ansatz gives an exact result for finite rank
- The ansatz corresponds to replica symmetry from the point of view of spin distributions (Panchenko)

Towards a Rigorous Result: Behavior of the Overlaps

Key Assumption: The bulk measure is equivalent to the posterior sample from the model

$$\boldsymbol{Y}'_{eff} = \sqrt{\sigma}\boldsymbol{X} + Z'_{eff} \in \mathbb{R}^{(N-1)\times M}.$$

- The ansatz gives an exact result for finite rank
- The ansatz corresponds to replica symmetry from the point of view of spin distributions (Panchenko)
- Overlap concentration no longer holds

Towards a Rigorous Result: Behavior of the Overlaps

Key Assumption: The bulk measure is equivalent to the posterior sample from the model

$$\boldsymbol{Y}'_{eff} = \sqrt{\sigma}\boldsymbol{X} + Z'_{eff} \in \mathbb{R}^{(N-1)\times M}.$$

- The ansatz gives an exact result for finite rank
- The ansatz corresponds to replica symmetry from the point of view of spin distributions (Panchenko)
- Overlap concentration no longer holds
- Failure of the Ghirlanda–Guerra identities

Summary

Rotationally Invariant I.I.D. Priors:

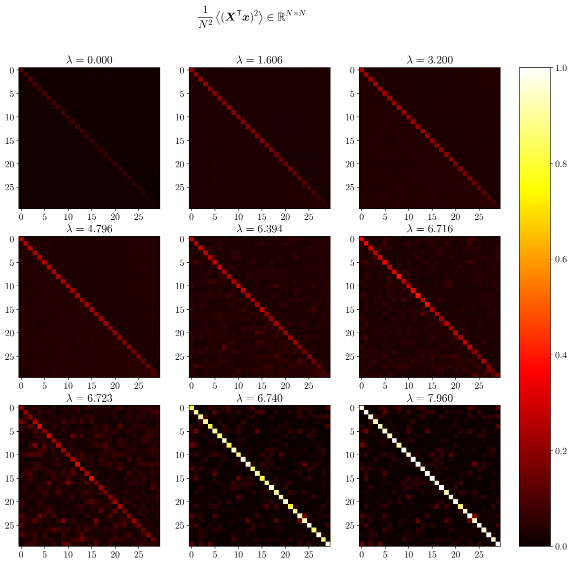
Finite Rank	**Sublinear Rank**	**Extensive Rank**
MI proved for all $M < \infty$	MI proved for $M = o(N)$	Explicit

Non-Rotationally Invariant I.I.D. Priors:

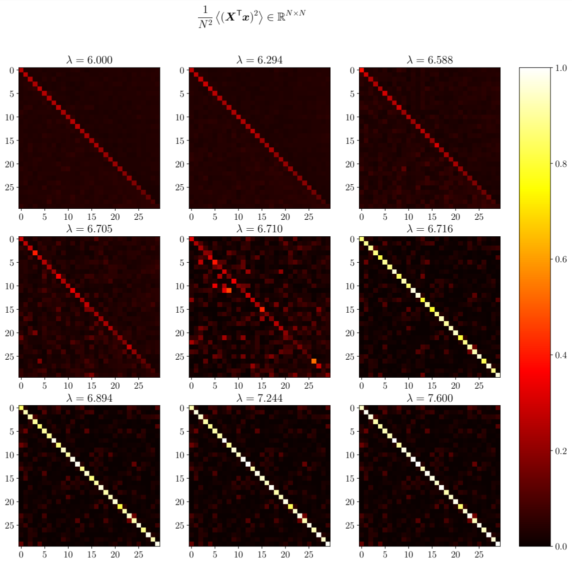
Finite Rank	**Sublinear Rank**	**Extensive Rank**
MI proved for all $M < \infty$	MI proved for $M = \log N$	Predictions

Thank you!

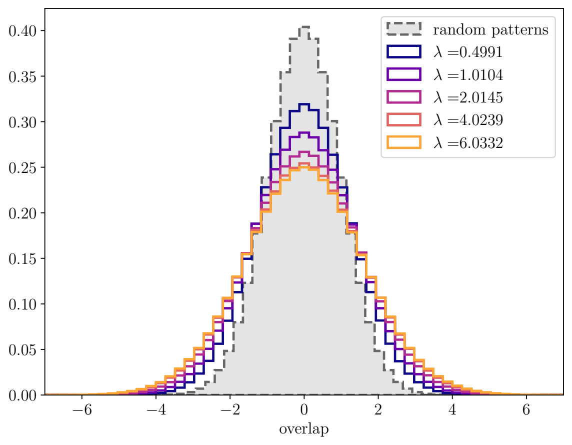
Numerical Results: Behavior of the Overlaps



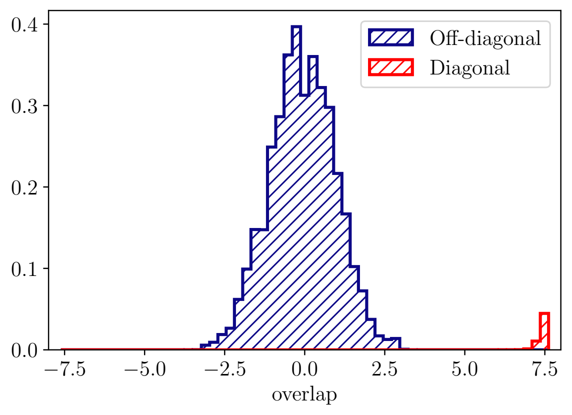
Numerical Results: Behavior of the Overlaps



Numerical Results: Entries of the Overlaps



Numerical Results: Entries of the Overlaps



Conjecture: Overlap Structure

Denoising Phase:

- Diagonal and offdiagonals are on the same order

Factorization Phase:

Conjecture: Overlap Structure

Denoising Phase:

- Diagonal and offdiagonals are on the same order
- Can only infer $\boldsymbol{X}\boldsymbol{X}^{\intercal}$.

Factorization Phase:

Conjecture: Overlap Structure

Denoising Phase:

- Diagonal and offdiagonals are on the same order
- Can only infer $\boldsymbol{X}\boldsymbol{X}^\intercal$.

Factorization Phase:

- Strong diagonals

Conjecture: Overlap Structure

Denoising Phase:

- Diagonal and offdiagonals are on the same order
- Can only infer $\boldsymbol{X}\boldsymbol{X}^\intercal$.

Factorization Phase:

- Strong diagonals
- Can infer $\boldsymbol{X}$.

Conjecture: Overlap Structure

Denoising Phase:

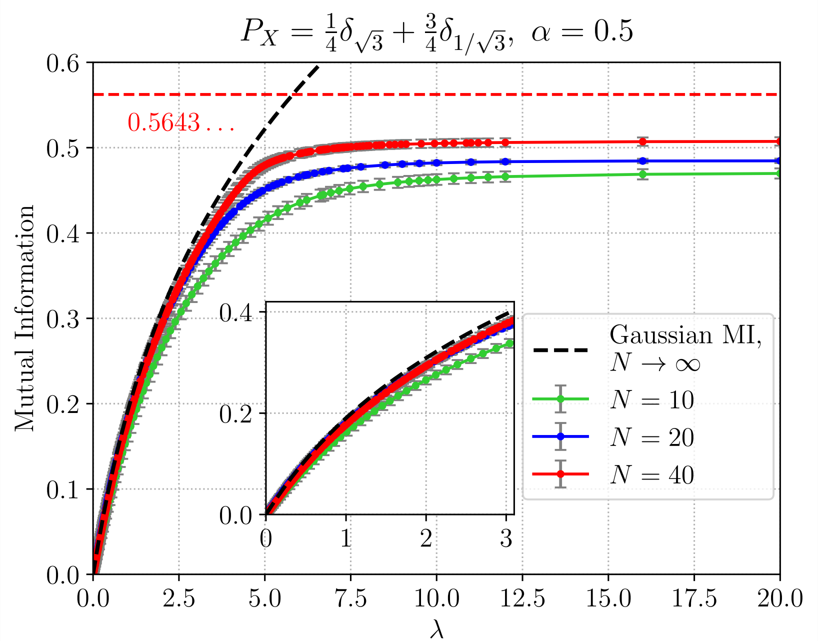
- Diagonal and offdiagonals are on the same order
- Can only infer $\boldsymbol{X}\boldsymbol{X}^\intercal$.

Factorization Phase:

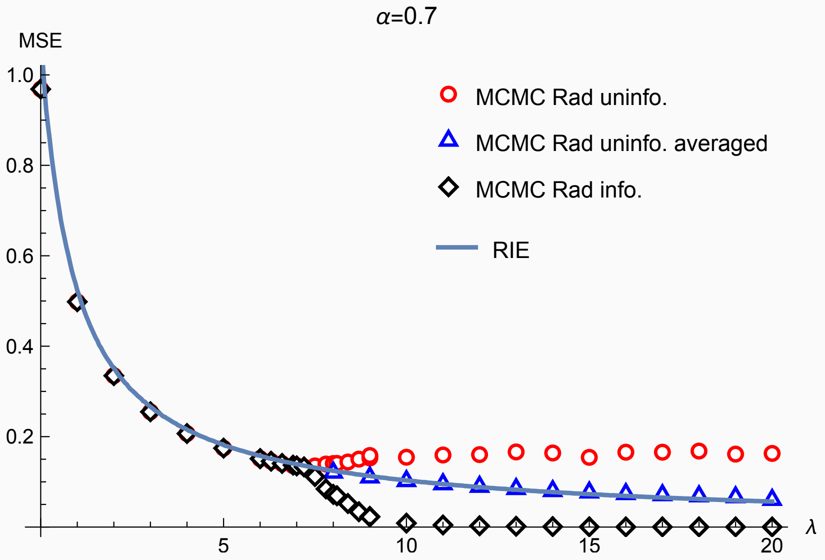
- Strong diagonals
- Can infer $\boldsymbol{X}$.

Goal: Prove that the overlaps have this asymptotic structure.

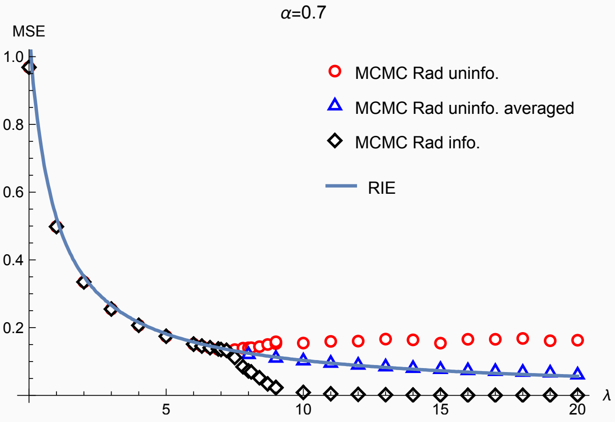
Numerical Results: Non-Rotational Invariant Prior



Algorithmic Phase Transition



Algorithmic Phase Transition



Statistical-to-Computational Gap: Need a strong informative initialization to beat the RIE