Spectral Analysis of q-Al-Salam–Carlitz Unitary Ensembles

Joint work with Sung-Soo Byun and Jaeseong Oh
arXiv: 2507.18042

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Contents

Spectral Analysis of Gaussian Unitary Ensemble

Gaussian unitary ensemble

$$\mathbf{H} = (h_{jk})_{j,k=1}^{N}$$

where

$$h_{jk} = \overline{h_{kj}} = \begin{cases} \dfrac{\xi_{jk} + i\eta_{jk}}{\sqrt{2}} & j \neq k \\ \xi_{jj} & j = k \end{cases}$$

with

$$\xi_{jk}, \eta_{jk} \sim \mathcal{N}(0, 1/N), \quad \text{i.i.d.}$$

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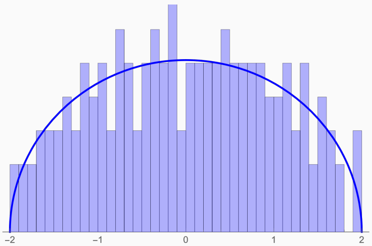

$N = 160$

Wigner Semi-Circle Law

Gaussian Unitary Ensemble

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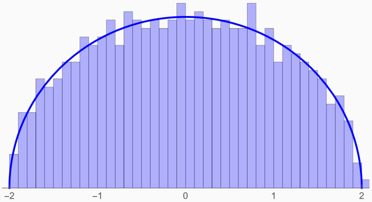

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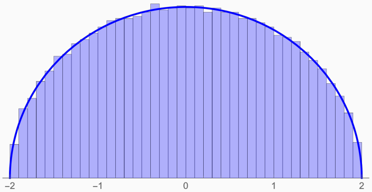

$N = 2560$

Wigner Semi-Circle Law

Wigner's Semi-Circle Law and Spectral Moments

- **Wigner's Semi-Circle Law (Wigner '55)**

$$\frac{1}{N}\sum_{j=1}^{N}\delta_{x_j/\sqrt{N}} \to d\mu_{\mathrm{sc}}(x) := \frac{\sqrt{4-x^2}}{2\pi}1_{[-2,2]}(x)\,dx$$

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- **Moments and Catalan Number**

$$\int_{-2}^{2} x^{2p}\,d\mu_{\mathrm{sc}}(x) = C_p := \frac{1}{p+1}\binom{2p}{p}$$

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- **Large-N Asymptotic of Spectral Moments**

$$\lim_{N\to\infty}\frac{1}{N^{p+1}}M_{N,2p}^{\mathrm{GUE}} = C_p$$

Wick's Formula & Non-crossing Pairing

$$\lim_{N\to\infty} \frac{1}{N^{p+1}} M_{N,2p}^{\mathrm{GUE}} = C_p$$

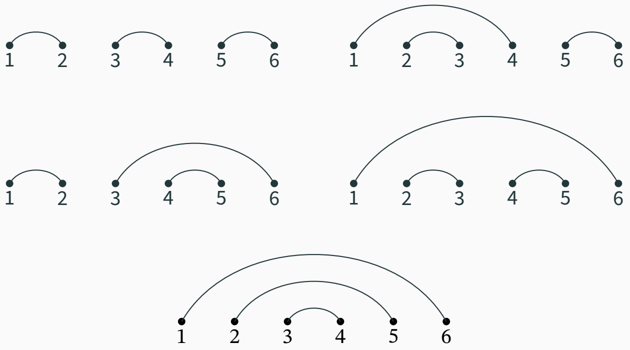

Figure 1: Non-crossing pairings on $2p = 6$

Harer-Zagier Formula for the GUE

$$(p+1)M_{N,2p}^{\mathrm{GUE}} = (4p-2)N\,M_{N,2p-2}^{\mathrm{GUE}} + (p-1)(2p-1)(2p-3)M_{N,2p-4}^{\mathrm{GUE}}$$

Harer-Zagier, *The Euler characteristic of the moduli space of curves*, Invent. Math. **85** (1986), 457–485.

Harer-Zagier Formula and Genus Expansion

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- **Genus Expansion**

$$M_{N,2p}^{\mathrm{GUE}} = \sum_{g=0}^{\lfloor p/2 \rfloor} c(g;p) N^{p+1-2g}$$

$c(g;p)$: # of pairings of edges of $2p$-gon, the resulting identification yielding a compact Riemann surface of genus g

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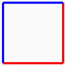
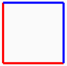
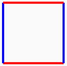
- **Example**: For $p = 2$,

$M_{N,2p} = \#\Big\{$  ,  $\Big\}N^3 + \#\Big\{$  $\Big\}N^{3-2}$

$$= 2N^3 + N$$

Integrable Structure of GUE

Joint PDF of Eigenvalues

$$\mathbb{P}_N^{\mathrm{GUE}}(\mathbf{x}) = \frac{1}{Z_N^{\mathrm{GUE}}} \prod_{1 \le j < k \le N} |x_j - x_k|^2 \prod_{l=1}^{N} e^{-\frac{1}{2}x^2}$$

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1-Point Function

$$\rho_N^{\mathrm{GUE}}(x) = N \int_{\mathbb{R}^{N-1}} \mathbb{P}_N^{\mathrm{GUE}}(x, x_2, \cdots, x_N)\, dx_2 \cdots dx_N$$

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- **Spectral Moment**

$$M_{N,p}^{\mathrm{GUE}} = \int_{\mathbb{R}} x^p \rho_N^{\mathrm{GUE}}(x)\, dx$$

Spectral Moment and Hermite Polynomials

- **Hermite Polynomial**

$$\int_{\mathbb{R}} \mathrm{He}_n(x)\mathrm{He}_m(x)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}\,dx = n!\delta_{n,m}$$

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- **Orthogonal Polynomial Representation**

$$\rho_N^{\mathrm{GUE}}(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}\sum_{j=0}^{N-1}\frac{1}{j!}\mathrm{He}_j(x)^2$$

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$$\int_{\mathbb{R}} x^p \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}\frac{1}{j!}\mathrm{He}_j(x)^2\,dx = (2p-1)!!\sum_{l=0}^{p}\binom{j}{l}\binom{p}{l}2^l$$

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- **Positive Sum Formula for Spectral Moments**

$$M_{N,2p}^{\mathrm{GUE}} = \int_{\mathbb{R}} x^{2p}\rho_N^{(\mathrm{GUE})}(x)\,dx = (2p-1)!!\sum_{l=0}^{p}\binom{N}{l+1}\binom{p}{l}2^l$$

Spectral Moments of Various Ensembles

- *Orthogonal Polynomial Ensemble*
  Cunden-Mezzadri-O'Connell-Simm '19, Gisonni-Grava-Ruzza' 21
- *Non-Hermitian Random Matrices*
  Forrester-Rains '09, Sommers-Khoruzhenko '09, Byun-Forrester '24, Byun '24, Akemann-Byun-Oh '25
- *High-dimensional Fermi Gas*
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- *Discrete Ensembles*
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Development and Applications of Spetral Moments

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Applications

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  Ginot-Gwilliam-Hamilton-Zeinalian '22
- *Time-delay Matrix of Quantum Dots & τ-function Theory*
  Livan-Vivo '11, Mezzadri-Simm '11–'13, Cunden '15, Cunden-Mezzadri-Simm-Vivo '16

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q-Deformed Gaussian Unitary Ensemble

q-Deformation

- **q-Analogues**: For $q \in (0, 1)$,

$$[n]_q := 1 + q + q^2 + \cdots + q^{n-1} = \frac{1 - q^n}{1 - q}$$

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$$
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[n]_q &:= 1 + q + q^2 + \cdots + q^{n-1} = \frac{1-q^n}{1-q} \\
[n]_q! &:= [n]_q \times [n-1]_q \times \cdots \times [1]_q \\
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- **q-Pochhammer Symbol**

$$(a;q)_n := \prod_{j=0}^{n-1}(1-aq^j) = (1-a)(1-aq)\cdots(1-aq^{n-1})$$

$$(a_1,\cdots,a_k;q)_\infty := \prod_{j=0}^{\infty}(1-a_1q^j)\cdots(1-a_kq^j)$$

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- **Jackson q-Integral**

$$\int_0^\alpha f(x)\,d_qx = (1-q)\sum_{k=0}^{\infty}\alpha q^k f(\alpha q^k), \quad \int_\alpha^\beta f(x)\,d_qx = \int_0^\beta f(x)\,d_qx - \int_0^\alpha f(x)\,d_qx$$

Discrete q-Hermite Polynomial

- **Hermite Polynomial Revisited**

$$\mathrm{He}_{n+1}(x) = x\mathrm{He}_n(x) - n\mathrm{He}_{n-1}(x)$$

with orthogonality

$$\int_{\mathbb{R}} \mathrm{He}_n(x)\mathrm{He}_m(x)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}\,dx = n!\delta_{n,m}$$

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Discrete q-Hermite Polynomial

$$H_{n+1}(x;q) = xH_n(x;q) - q^{n-1}(1-q^n)H_{n-1}(x;q)$$

with orthogonality

$$\int_{-1}^{1} H_n(x;q)H_m(x;q)\frac{(qx,-qx;q)_\infty}{(q,-1,-q;q)_\infty}\,d_qx = (1-q)(q;q)_n q^{\frac{n(n-1)}{2}}\delta_{nm}$$

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cf. Continuum Limit

$$\lim_{q\to 1}(1-q)^{-\frac{n}{2}}H_n\left(\sqrt{1-q}\,x;q\right) = \mathrm{He}_n(x)$$

Joint PDF of the GUE Revisited

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Joint PDF of the q-Deformed GUE

$$\mathbb{P}_N^{\mathrm{qGUE}}(\mathbf{x}) = \frac{1}{Z_N^{\mathrm{qGUE}}} \prod_{1\le j<k\le N} \left|x_j - x_k\right|^2 \prod_{l=1}^{N} \frac{(qx_l, -qx_l; q)_\infty}{(q, -1, -q; q)_\infty}$$

supported on the q-lattice $\pm q^{\mathbb{Z}_{\ge 0}}$

- **Joint PDF of the GUE Revisited**

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supported on the q-lattice $\pm q^{\mathbb{Z}_{\ge 0}}$

- **1-Point Function of the q-deformed GUE**

$$\rho_N^{\mathrm{qGUE}}(x; q) = \frac{1}{1-q} \frac{(qx, -qx; q)_\infty}{(q, -1, -q; q)_\infty} \sum_{j=0}^{N-1} \frac{1}{(q; q)_j q^{\frac{j(j-1)}{2}}} H_j(x; q)^2$$

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$$M_{N,2p}^{\mathrm{GUE}} = \int_{\mathbb{R}} x^{2p} \rho_N^{\mathrm{GUE}}(x)\, dx = (2p-1)!! \sum_{l=0}^{p} \binom{N}{l+1} \binom{p}{l} 2^l$$

Spectral Moments of the q-Deformed GUE

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- **Spectral Moments of the q-Deformed GUE (Byun-Forrester-Oh '24)**

$$\begin{aligned} M_{N,2p}^{\mathrm{qGUE}} &= \int_{-1}^{1} x^{2p} \rho_N^{\mathrm{qGUE}}(x;q)\, d_q x \\ &= (1-q)^p \sum_{j=0}^{N-1} \sum_{l=0}^{p} q^{(j-l)(2p-l)+\frac{l(l-1)}{2}} \begin{bmatrix} j \\ l \end{bmatrix}_q \frac{[2p]_q!}{[2p-2l]_q!![l]_q!} \end{aligned}$$

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Spectral Moments of the q-Deformed GUE (Byun-Forrester-Oh ’24)

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cf. Alternating Sum Formula (based on the Schur polynomial average)

Forrester-Li-Shen-Yu, *q-Pearson pair and moments in q-deformed ensembles*, Ramanujan J. **60** (2023), 195–235.

Spectral Moments of the q-Deformed GUE

Spectral Moments of the GUE Revisited

$$M_{N,2p}^{\mathrm{GUE}} = \int_{\mathbb{R}} x^{2p} \rho_N^{\mathrm{GUE}}(x)\, dx = (2p-1)!! \sum_{l=0}^{p} \binom{N}{l+1}\binom{p}{l} 2^l$$

Spectral Moments of the q-Deformed GUE (Byun-Forrester-Oh '24)

$$M_{N,2p}^{\mathrm{qGUE}} = \int_{-1}^{1} x^{2p} \rho_N^{\mathrm{qGUE}}(x;q)\, d_q x$$
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cf. Continuum Limit

$$\lim_{q\to 1} (1-q)^{-p} M_{N,2p}^{\mathrm{qGUE}} = M_{N,2p}^{\mathrm{GUE}}$$

Genus-Type Expansion of the q-Deformed GUE

- **Genus Expansion Revisited**

$$\frac{1}{N^p} M_{N,2p}^{\mathrm{GUE}} = c(0;p)N + c(1;p)\frac{1}{N} + O\Big(\frac{1}{N^3}\Big)$$

with $c(0;p) = C_p$

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- **Double Scaling**

$$q = e^{-\frac{\lambda}{N}}, \qquad \lambda > 0$$

Forrester, *Global and local scaling limits for the $\beta = 2$ Stieltjes-Wigert random matrix ensemble*, Random Matrices Theory Appl. **11** (2022), 2250020.

Husson-Mazzuca-Occelli, *Discrete and Continuous Muttalib–Borodin process: Large deviations and Riemann–Hilbert analysis*, arXiv: 2025.23164

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Cf. Continuum Limit

$$\lim_{\lambda \to 0} \frac{1}{\lambda^p} \mathcal{M}_{2p,0}^{\mathrm{qGUE}} = C_p$$

- **Wigner's Semi-Circle Law Revisited**

$$\frac{1}{\sqrt{N}}\rho_N(\sqrt{N}x)\,dx \Longrightarrow d\mu_{\mathrm{sc}} = \frac{\sqrt{4-x^2}}{2\pi}1_{[-2,2]}(x)\,dx$$

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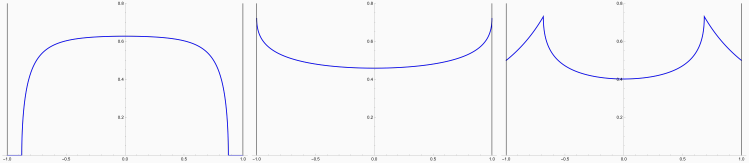
Limiting Density of the q-Deformed GUE (Byun-Forrester-Oh '24)

$$\frac{1}{N}\rho_N^{\mathrm{qGUE}}(x;q)\,dx\Big|_{q=e^{-\lambda/N}} \Longrightarrow \rho^{\mathrm{qGUE}}(x)\,dx$$



(a) $\lambda < \lambda_c$ **(b)** $\lambda = \lambda_c$ **(c)** $\lambda > \lambda_c$

Limiting Density of the q-Deformed GUE

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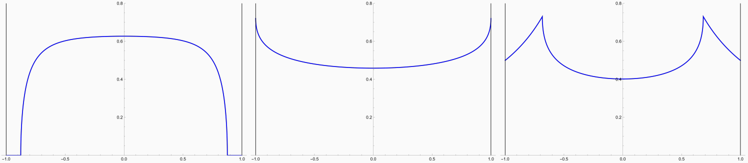
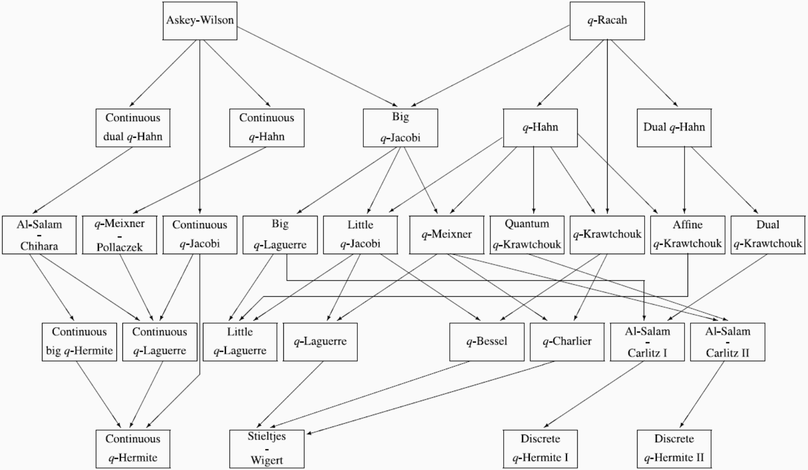


(a) $\lambda < \lambda_c$ **(b)** $\lambda = \lambda_c$ **(c)** $\lambda > \lambda_c$

cf. q-deformed Wigner's Semi-circle Law ($\lambda \to 0$)

Askey Scheme



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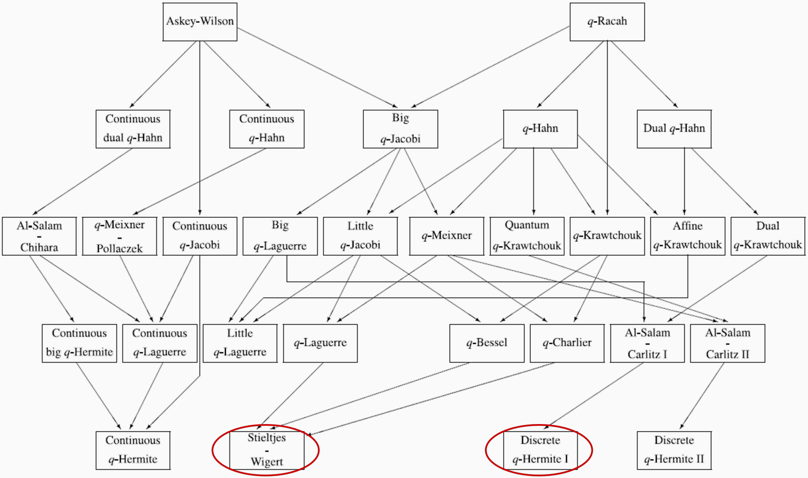

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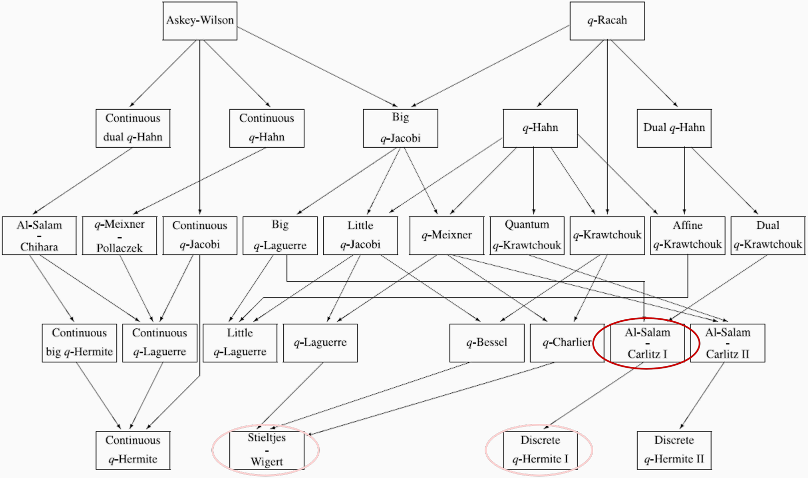

q-Al-Salam–Carlitz Unitary Ensemble

■ **Discrete q-Hermite Polynomial Revisited**

$$H_{n+1}(x;q) = xH_n(x;q) - q^{n-1}(1-q^n)H_{n-1}(x;q)$$

with orthogonality

$$\int_{-1}^{1} H_n(x;q)H_m(x;q)\frac{(qx,-qx;q)_\infty}{(q,-1,-q;q)_\infty}\,d_qx = (1-q)(q;q)_n q^{\frac{n(n-1)}{2}}\delta_{nm}$$

Al-Salam–Carlitz Polynomial

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- **Al-Salam–Carlitz Polynomial** For $a<0$,

$$U_{n+1}^{(a)}(x;q) = (x-(a+1)q^n)U_n^{(a)}(x;q) + aq^{n-1}(1-q^n)U_{n-1}^{(a)}(x;q)$$

with orthogonality

$$\int_{a}^{1} U_n^{(a)}(x;q)U_m^{(a)}(x;q)\frac{(qx,qx/a;q)_\infty}{(q,a,q/a;q)_\infty}\, d_qx = (-a)^n(1-q)q^{\frac{n(n-1)}{2}}\delta_{nm}$$

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cf. $H_n(x;q) = U_n^{(-1)}(x;q)$

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- **Continuum Limits**
  - Hermite polynomial

$$\frac{1}{2^n}H_n(x-r) = \lim_{q\uparrow 1}\frac{U_n^{(a)}(x;q)}{(1-q^2)^{n/2}}\Bigg|_{x\mapsto x\sqrt{1-q^2},\,a\mapsto r\sqrt{1-q^2}-1}$$

  - Charlier polynomial

$$a^nC_n(x;a) = \lim_{q\uparrow 1}\frac{U_n^{(a)}(x;q)}{(1-q)^n}\Bigg|_{x\mapsto q^x,\,a\mapsto a(q-1)}$$

Al-Salam–Carlitz Unitary Ensemble

- **1-Point Function of the q-deformed GUE Revisited**

$$\rho_N^{\mathrm{qGUE}}(x;q) = \frac{1}{1-q}\frac{(qx,-qx;q)_\infty}{(q,-1,-q;q)_\infty}\sum_{j=0}^{N-1}\frac{1}{(q;q)_j q^{\frac{j(j-1)}{2}}}H_j(x;q)^2$$

supported on the q-lattice $\pm q^{\mathbb{Z}_{\geq 0}}$

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- **Spectral Moment of the Al-Salam–Carlitz UE**

$$M_{N,p}^{(a,q)} = \int_a^1 x^p \rho_N^{(a)}(x;q)\, d_q x$$

Spectral Moments of Al-Salam–Carlitz UE

Theorem 1 (Byun-J.-Oh '25)

For any $p, N \in \mathbb{N}$ and $a < 0$,

$$M_{N,p}^{(a,q)} = \sum_{j=0}^{N-1} \sum_{k=0}^{\lfloor p/2 \rfloor} \frac{(-a)^k (1-q)^k}{(a+1)^{2k-p}} \sum_{l=0}^{k} \frac{q^{-l(p-l)+\frac{l(l-1)}{2}} [p]_q!}{[p-2l]_q!! \, [l]_q!} \mathrm{H}(k-l, p-2k) q^{j(p-l)} \begin{bmatrix} j \\ l \end{bmatrix}_q$$

where

$$\mathrm{H}(b,c) := \sum_{0 \le j_1 \le j_2 \le \cdots \le j_c \le b} \prod_{l=1}^{c} \frac{[2j_j + l - 2]_q!!}{[2j_l + l - 1]_q!!}$$

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- **Alternating Sum Formula** (based on the MacDonald polynomial and superintegrability identity)

  Byun-Forrester, *On the superintegrability of the Gaussian β ensemble and its (q,t) generalisation*, arXiv:2505.12927.

- **Symmetry of Spectral Moment**

$$M_{N,p}^{(1/a,q)} = \frac{1}{a^p} M_{N,p}^{(a,q)}$$

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- **Examples**

$$M_{N,0}^{(a,q)} = N$$

$$M_{N,1}^{(a,q)} = (a+1) \frac{1-q^N}{1-q}$$

$$M_{N,2}^{(a,q)} = \frac{1-q^N}{q(1-q)^2} \Big((a^2+1)q + q^N (q + a(1 + 2q + q^2 + aq)) \Big)$$

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- **Al-Salam–Carlitz Integral**

$$\int_a^1 x^p \frac{(qx, qx/a; q)_\infty}{(q, a, q/a; q)_\infty} \frac{1}{(-a)^j (q;q)_j q^{j(j-1)/2}} U_j^{(a)}(x;q)^2 \, d_q x$$

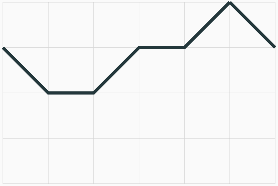
Three-Term Recurrence of Orthogonal Polynomials

$$P_{n+1}(x) = (x - b_n)P_n(x) - \lambda_n P_{n-1}(x)$$

- **Three-Term Recurrence of Orthogonal Polynomials**

$$P_{n+1}(x) = (x - b_n)P_n(x) - \lambda_n P_{n-1}(x)$$

- **Motzkin Path**



Combinatorics of Orthogonal Polynomials

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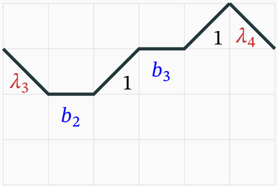

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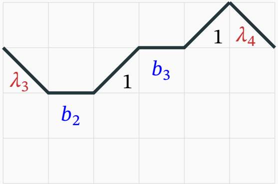

- **Flajolet-Viennot Theory**

$$\int_{\mathbb{R}} x^p \frac{P_j(x)^2}{h_j} w(x)\,dx = \sum_{\Gamma:(0,j)\to(p,j)} \mathrm{wt}(\Gamma)$$

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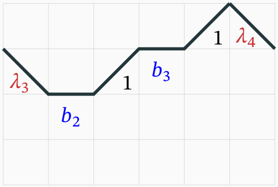

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cf. Spectral Moment as a Partition Function

Bryc-Kuznetsov-Wesolowski, *Limits of random Motzkin paths with KPZ related asymptotics*, Int. Math. Res. Not. **2025** (2025), 1-33.

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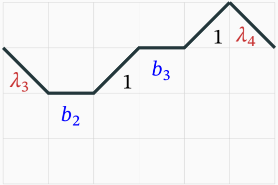

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cf. Evaluation from Combinatorics for $j = 0$

Corteel-Jonnadula-Keating-Kim, *Lecture hall graphs and the Askey scheme*, arXiv:2311.12761.

- **Matching**

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- **Statistics of Matching**

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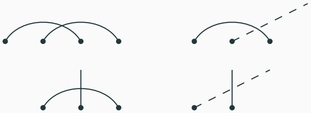
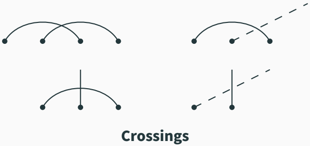


Crossings

Matching Problem

- **Matching**

- **Statistics of Matching**



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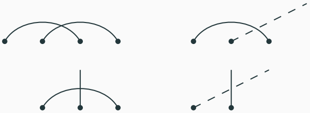
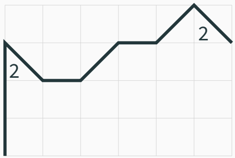


Crossings

Nestings

- **Al-Salam–Carlitz History**



- **Example:** $\Gamma : (0,3) \to (6,3)$

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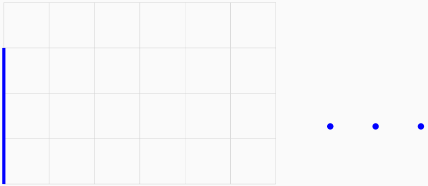
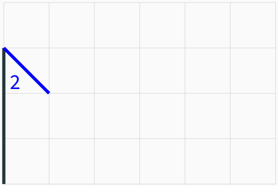


Double Counting

- **Example:** $\Gamma : (0,3) \to (6,3)$



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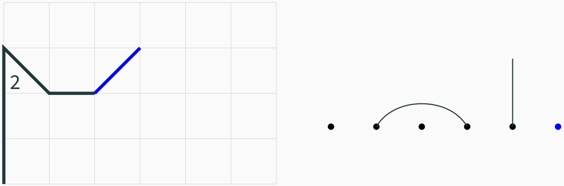

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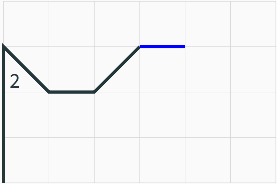

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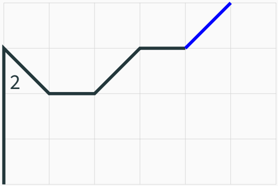

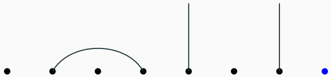

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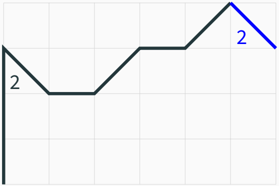

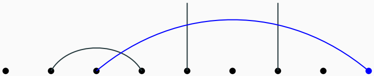

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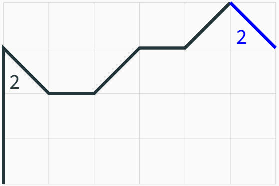

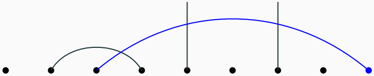

Lemma (Byun-J.-Oh '25)

$$\mathrm{wt}(\Gamma) = \sum_{M} q^{\#\mathrm{Crossing}(M)+2\#\mathrm{Nesting}(M)}$$

where the sum runs over matchings corresponded to all Al-Salam–Carlitz histories on Γ.

Spectral Moments of Al-Salam–Carlitz UE

- **Al-Salam–Carlitz Integral (Byun-J.-Oh '25)**

$$\int_a^1 x^p \frac{(qx, qx/a; q)_\infty}{(q, a, q/a; q)_\infty} \frac{1}{(-a)^j (q;q)_j q^{j(j-1)/2}} U_j^{(a)}(x;q)^2 \, d_q x$$

$$= \sum_{k=0}^{\lfloor p/2 \rfloor} \frac{(-a)^k (1-q)^k}{(a+1)^{2k-p}} \sum_{l=0}^{k} \frac{q^{-l(p-l)+\frac{l(l-1)}{2}} [p]_q!}{[p-2l]_q!! \, [l]_q!} \mathsf{H}(k-l, p-2k) q^{j(p-l)} \begin{bmatrix} j \\ l \end{bmatrix}_q$$

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Theorem 1 (Byun-J.-Oh ’25)

For any $p, N \in \mathbb{N}$ and $a < 0$,

$$M_{N,p}^{(a,q)} = \sum_{j=0}^{N-1} \sum_{k=0}^{\lfloor p/2 \rfloor} \frac{(-a)^k (1-q)^k}{(a+1)^{2k-p}} \sum_{l=0}^{k} \frac{q^{-l(p-l)+\frac{l(l-1)}{2}} [p]_q!}{[p-2l]_q!! \, [l]_q!} \mathsf{H}(k-l, p-2k) q^{j(p-l)} \begin{bmatrix} j \\ l \end{bmatrix}_q$$

where

$$\mathsf{H}(b,c) := \sum_{0 \le j_1 \le j_2 \le \cdots \le j_c \le b} \prod_{l=1}^{c} \frac{[2j_j + l - 2]_q!!}{[2j_l + l - 1]_q!!}$$

Large N-expansion of Spetral Moments

Theorem 2 (Byun-J.-Oh '25)

With $q = e^{-\lambda/N}$ $(\lambda > 0)$, as $N \to \infty$,

$$q^{\frac{p}{2}} M_{N,p}^{(a,q)} = \mathcal{M}_{p,0} N + \frac{\mathcal{M}_{p,1}}{N} + O(N^{-3})$$

where

$$\mathcal{M}_{p,0} = \frac{1}{\lambda} \sum_{l=0}^{\lfloor p/2 \rfloor} (a+1)^{p-2l} (-a)^l \frac{(p-l-1)!}{l!(p-2l)!} I_{1-e^{-\lambda}}(l+1, p-l).$$

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Continuum Limit

$$\lim_{\lambda \to 0} \frac{1}{\lambda^{p/2}} \mathcal{M}_{p,0} \Big|_{a=-1+r\sqrt{\lambda}} = \sum_{l=0}^{\lfloor p/2 \rfloor} \binom{p}{2l} r^{p-2l} C_l$$

cf. $U_n^{(a)}(x) \to H_n(x-r)$ with proper scaling

Limiting Spectral Density

Theorem 3 (Byun-J.-Oh '25)

With $q = e^{-\lambda/N}$ $(\lambda > 0)$, as $N \to \infty$,

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- **(Support of $\rho^{(a)}$)** For $-1 \le a < 0$,

$$\mathrm{supp}(\rho^{(a)}(x)) = \begin{cases} (\mathsf{u}-\mathsf{v}, \mathsf{u}+\mathsf{v}) & \text{if } \lambda \in (0, \lambda_c^{(1,a)}), \\ (\mathsf{u}-\mathsf{v}, 1) & \text{if } \lambda \in (\lambda_c^{(1,a)}, \lambda_c^{(2,a)}), \\ (a, 1) & \text{if } \lambda \in (\lambda_c^{(2,a)}, \infty). \end{cases}$$

where

$$\mathsf{u} \equiv \mathsf{u}(\lambda) := (1+a)e^{-\lambda}, \qquad \mathsf{v} \equiv \mathsf{v}(\lambda) := 2\sqrt{-a(1-e^{-\lambda})e^{-\lambda}}$$

Limiting Spectral Density

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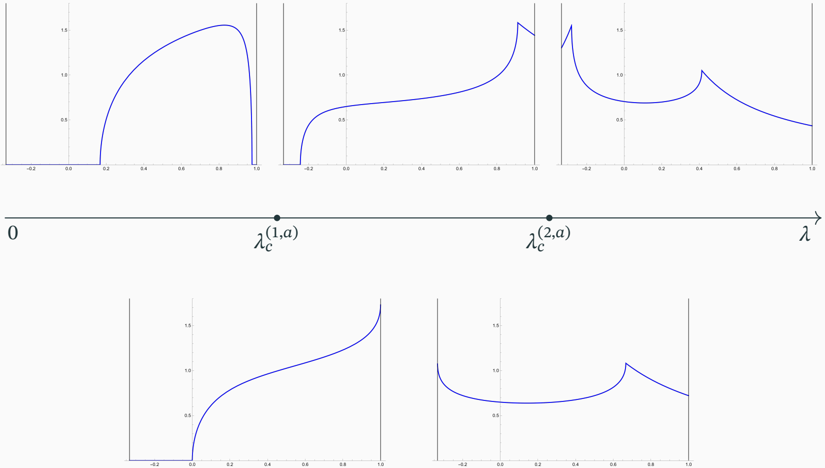
- **(Density)**

$$\rho^{(a)}(x) = \frac{2}{\pi\lambda|x|} \arctan\sqrt{\frac{1-x_0-x_1}{1-x_0+x_1}\frac{1-e^{-\lambda}-x_0+x_1}{x_0+x_1-1+e^{-\lambda}}}\, 1_{(\mathrm{u-v,u+v})}(x)$$
$$+\begin{cases} 0 & \text{if } \lambda \in (0, \lambda_c^{(1,a)}), \\ \frac{1}{\lambda|x|}\, 1_{(\mathrm{u+v},1)}(x) & \text{if } \lambda \in (\lambda_c^{(1,a)}, \lambda_c^{(2,a)}), \\ \frac{1}{\lambda|x|}\, 1_{(a,\mathrm{u-v})\cup(\mathrm{u+v},1)}(x) & \text{if } \lambda \in (\lambda_c^{(2,a)}, \infty). \end{cases}$$

where

$$x_0 = \frac{a^2+1-x(a+1)}{(a-1)^2}, \qquad x_1 = \frac{\sqrt{4a(x-a)(x-1)}}{(a-1)^2}.$$

Limiting Spectral Density



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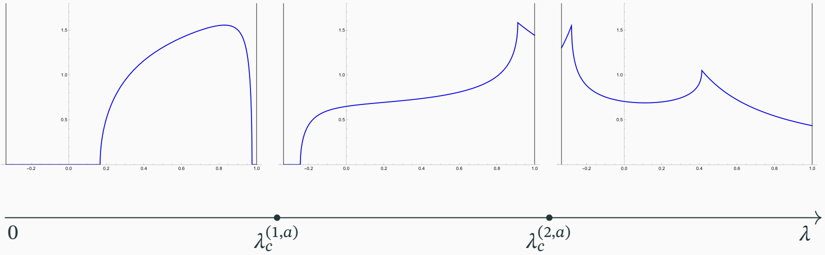

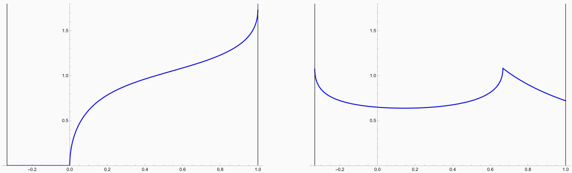

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$$\lim_{\lambda\to 0}\sqrt{\lambda}\rho^{(a)}(\sqrt{\lambda}x)\Big|_{a=-1+r\sqrt{\lambda}} = \mu_{\mathrm{sc}}(x-r)$$

Limiting Zero Distribution and Limiting Density

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$$\nu(P_n) := \frac{1}{n}\sum_{j=1}^{n} \delta_{x_{N,j}} \Longrightarrow \mu$$

where $x_{N,1}, \cdots, x_{N,N}$ are zeros of P_N.

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cf. Viewpoint from Finite Free Probability

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- **Limiting Zero Distribution of Al-Salam–Carlitz Polynomial (Byun-J.-Oh ’25)**

$$\nu\big(U_n^{(a)}(x;q)\big)\Big|_{q=e^{-\frac{\lambda}{n}}} \Longrightarrow \rho^{(a)}$$

- **Arcsine Measure**

$$\frac{d\omega_{[\alpha,\beta]}(t)}{dt} = \begin{cases} \dfrac{1}{\pi\sqrt{(\beta - t)(t-\alpha)}} & \text{if } t \in (\alpha, \beta) \\ 0 & \text{otherwise} \end{cases}$$

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- **Three-Term Recurrence of Orthonormal Polynomials**

$$xP_{n,N}(x) = a_{n+1,N}P_{n+1,N}(x) + b_{n,N}P_{n,N}(x) + a_{n,N}P_{n-1,N}(x)$$

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Limiting Spectral Density and Zero Distribution

Discrete Unitary Invariant Ensemble

↓

q-Orthogonal Polynomial

↓ ↓

Spectral Moment | **Empirical Zero Distribution**

↓ ↓

Limiting Spectral Density ⟷ **Limiting Zero Distribution**

Future Directions

- **Spectral Analysis of q-deformed LUE**

  Forrester-Li-Shen-Yu, *q-Pearson pair and moments in q-deformed ensembles*, Ramanujan J. **60** (2023), 195–235.

- **Spectral Moment Formula for q-deformed GO/SE**

  Li–Shen–Yu-Forrester, *Discrete orthogonal ensemble on the exponential lattices*, Adv. Appl. Math. **164** (2025), 102836.

  Forrester–Li, *Classical discrete symplectic ensembles on the linear and exponential lattice: skew orthogonal polynomials and correlation functions*, Trans. Amer. Math. Soc. **373** (2020), 665–698.

- **Asymptotic Behaviour of q-Hermite Polynomial and Local Statistics**

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Thank you!