Cumulant structures of entanglement entropy over Hilbert-Schmidt ensemble

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Entanglement Estimation

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- Task: estimate the degree of entanglement of **quantum bipartite model*** measured by von Neumann entropy over Hilbert-Schmidt ensemble

*Page [1993] Average entropy of a subsystem, *Phys. Rev. Lett.*

Bipartite Model

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- Generic state of two subsystems A and B of Hilbert space dimensions m and n

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- Density matrix

$$\rho = |\psi\rangle \langle\psi| , \qquad \mathrm{tr}(\rho) = 1$$

- Bipartite model is obtained by partial trace (purification) of ρ leading to a reduced density matrix

$$\rho_A = \mathrm{tr}_B(\rho)$$

Ensemble and Entropy

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- Degree of entanglement is encoded in the cumulants of entropy $\kappa_l(S)$

Ensemble and Entropy

Computing the first l cumulants of S can be converted to the first l cumulants of induced entropy

$$T = \sum_{i=1}^{m} x_i \ln x_i$$

over the Laguerre unitary ensemble

$$\prod_{1 \leq i < j \leq m} (x_i - x_j)^2 \prod_{i=1}^{m} w(x_i)$$

where

$$w(x) = x^{\alpha} \mathrm{e}^{-x}, \qquad \alpha = n - m$$

Preliminary

The l-th cumulant $\kappa_l(X)$ of a linear statistics

$$X = \sum_{i=1}^{m} f(x_i)$$

over a determinantal point process is given by*

$$\kappa_l(X) = \sum_{i=1}^{l} \mathrm{I}_i$$

where

$$\mathrm{I}_i = \sum_{l_1+\cdots+l_i=l} \frac{(-1)^{i-1}}{i} \frac{l!}{l_1!\cdots l_i!} \int \prod_{j=1}^{i} f^{l_j}(x_j)\, K(x_j, x_{j+1}) \mathrm{d}x_j$$

and $K(x_j, x_{j+1})$ is the correlation kernel with $x_{i+1} = x_1$

*Soshnikov [2002] Gaussian limit for determinantal random point fields, *Ann. Probab.*

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- $\kappa(T) = \mathrm{I}_1$

$$\mathrm{I}_1 = \int_0^\infty x \ln x \; K(x,x) \mathrm{d}x$$

- $\kappa_2(T) = \mathrm{I}_1 - \mathrm{I}_2$

$$\begin{aligned} \mathrm{I}_1 &= \int_0^\infty x^2 \ln^2 x \; K(x,x) \mathrm{d}x \\ \mathrm{I}_2 &= \int_0^\infty \int_0^\infty xy \ln x \ln y \; K(x,y) K(y,x) \mathrm{d}x \mathrm{d}y \end{aligned}$$

Preliminary

- $\kappa_3(T) = \mathrm{I}_1 - 3\mathrm{I}_2 + 2\mathrm{I}_3$

$$\begin{aligned}
\mathrm{I}_1 &= \int_0^\infty x^3 \ln^3 x \; K(x,x)\,\mathrm{d}x \\
\mathrm{I}_2 &= \int_0^\infty \int_0^\infty x^2 y \ln^2 x \ln y \; K(x,y)K(y,x)\mathrm{d}x\mathrm{d}y \\
\mathrm{I}_3 &= \int_0^\infty \int_0^\infty \int_0^\infty xyz \ln x \ln y \ln z \; K(x,y)K(y,z)K(z,x)\mathrm{d}x\mathrm{d}y\mathrm{d}z
\end{aligned}$$

Existing Methods and Results

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To obtain the l-th cumulant $\kappa_l(T)$, each integral $\mathrm{I}_i, \; i = 1, \ldots, l$, is explicitly computed using the following three steps

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- **1. Decouple.** Replacing every $K(x, y)$ in the integrals I_i with the summation form of Laguerre kernel

$$K(x, y) = \sqrt{w(x)w(y)} \sum_{k=0}^{m-1} \frac{k!}{(k+\alpha)!} L_k^{(\alpha)}(x) L_k^{(\alpha)}(y)$$

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- **2. Compute.** Using up to l derivatives (w.r.t. q) of

$$\int_0^\infty x^q \mathrm{e}^{-x} L_s^{(\alpha)}(x) L_t^{(\beta)}(x) \, \mathrm{d}x$$
$$= \quad (-1)^{s+t} \sum_{k=0}^{\min(s,t)} \binom{q-\alpha}{s-k} \binom{q-\beta}{t-k} \frac{\Gamma(q+1+k)}{k!}$$

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- **3. Simplify.** The bulk of calculation lies in the simplification of resulting i-nested sums in each I_i, which is an increasingly tedious and case-by-case task for higher-order cumulants

Results by Existing Methods

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- **Mean**: conjectured by Page'93*, proved in Foong-Kanno'94†, Sánchez-Ruiz'95‡ (among other proofs)

$$\kappa(S) = \psi_0(mn + 1) - \psi_0(n) - \frac{m + 1}{2n}$$

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- **Variance:** conjectured by Vivo-Pato-Oshanin'16§, proved in Wei'17¶

$$\kappa_2(S) = -\psi_1(mn+1) + \frac{m+n}{mn+1}\psi_1(n) - \frac{(m+1)(m+2n+1)}{4n^2(mn+1)}$$

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§Vivo-Pato-Oshanin [2016] Random pure states: Quantifying bipartite entanglement beyond the linear statistics, *Phys. Rev. E*

¶Wei [2017] Proof of Vivo-Pato-Oshanin's conjecture on the fluctuation of von Neumann entropy, *Phys. Rev. E*

Results by Existing Methods

- **Skewness*** and **kurtosis**† are also available

*Wei [2020] Skewness of von Neumann entanglement entropy, *J. Phys. A*

†Huang-Wei-Collaku [2021] Kurtosis of von Neumann entanglement entropy, *J. Phys. A*

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Summary of the first four cumulants over HS ensemble:

$$
\begin{aligned}
\kappa_1 &= a_1\psi_0(mn+1) + a_2\psi_0(n) + a_3 \\
\kappa_2 &= b_1\psi_1(mn+1) + b_2\psi_1(n) + b_3 \\
\kappa_3 &= c_1\psi_2(mn+1) + c_2\psi_2(n) + c_3\psi_1(n) + c_4 \\
\kappa_4 &= d_1\psi_3(mn+1) + d_2\psi_3(n) + d_3\psi_2(n) + d_4\psi_1^2(n) + d_5\psi_1(n) + d_6
\end{aligned}
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Example: cancellations in $\kappa_2(T)$ calculation over HS ensemble

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$$\begin{aligned}\mathrm{I}_1 \;=\; & a_1 + a_2\psi_0(n) + a_3\psi_0(n-m) + a_4\left(\psi_0(n) - \psi_0(m) + \psi_0(1)\right) \times \\ & \psi_0(n-m) + a_5\left(\psi_0^2(n-m) - \psi_1(n-m)\right) + \\ & a_6\sum_{k=1}^{m}\frac{\psi_0(k+n-m)}{k}\end{aligned}$$

$$\begin{aligned}\mathrm{I}_2 \;=\; & b_1 + b_2\psi_0(n) + b_3\psi_0(n-m) + b_4\psi_0^2(n) + b_5(\psi_0(n) - \psi_0(m) + \\ & \psi_0(1))\psi_0(n-m) + b_6\left(\psi_0^2(n-m) + \psi_1(n) - \psi_1(n-m)\right) + \\ & b_7\sum_{k=1}^{m}\frac{\psi_0(k+n-m)}{k}\end{aligned}$$

Anomalies in κ_4 Calculation over HS Ensemble

$\Omega_1 = \sum_{k=1}^{m} \frac{\psi_0(k+\alpha)}{k}$	$\Omega_6 = \sum_{k=1}^{m} \frac{\psi_0(k)\psi_0(k+\alpha)}{k}$	$\Omega_{11} = \sum_{k=1}^{m} \frac{\psi_1(k+\alpha)}{k+\alpha}$
$\Omega_2 = \sum_{k=1}^{m} \frac{\psi_0(k+\alpha)}{k^2}$	$\Omega_7 = \sum_{k=1}^{m} \frac{\psi_0^3(k+\alpha)}{k}$	$\Omega_{12} = \sum_{k=1}^{m} \frac{\psi_0(k)\psi_1(k+\alpha)}{k}$
$\Omega_3 = \sum_{k=1}^{m} \frac{\psi_0^2(k+\alpha)}{k}$	$\Omega_8 = \sum_{k=1}^{m} \frac{\psi_0^3(k+\alpha)}{k+\alpha}$	$\Omega_{13} = \sum_{k=1}^{m} \frac{\psi_0(k+\alpha)\psi_1(k+\alpha)}{k}$
$\Omega_4 = \sum_{k=1}^{m} \frac{\psi_0^2(k+\alpha)}{k+\alpha}$	$\Omega_9 = \sum_{k=1}^{m} \frac{\psi_0(k)\psi_0^2(k+\alpha)}{k}$	$\Omega_{14} = \sum_{k=1}^{m} \frac{\psi_2(k+\alpha)}{k}$
$\Omega_5 = \sum_{k=1}^{m} \frac{\psi_0^2(k+\alpha)}{k^2}$	$\Omega_{10} = \sum_{k=1}^{m} \frac{\psi_1(k+\alpha)}{k}$	$\Omega_{15} = \sum_{k=1}^{m} \frac{\psi_2(k+\alpha)}{k+\alpha}$

New Methods*

*Huang-Wei [2025] Cumulant structures of entanglement entropy, available at arXiv:2502.05371

Cumulant Structures: The Example of $\kappa_2(T)$

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Define

$$R_k = \sum_{i=1}^{m} x_i^k, \qquad T_k = \sum_{i=1}^{m} x_i^k \ln x_i$$

To find the cumulant

$$\kappa\left(T_k, T\right) = \mathrm{I}_1 - \mathrm{I}_2,$$

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we construct a related cumulant

$$\kappa\left(T_{k+1}, T_0\right) = \mathrm{I}_1 - \widetilde{\mathrm{I}}_2$$

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$$\begin{aligned}
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\widetilde{\mathrm{I}}_2 &= \int_0^\infty \int_0^\infty x^{k+1} \ln x \ln y \; K(x,y) K(y,x) \mathrm{d}x \mathrm{d}y
\end{aligned}$$

Cumulant Structures: The Example of $\kappa_2(T)$

To find the cumulant

$$\kappa\,(T_k,\, T) = \mathrm{I}_1 - \mathrm{I}_2$$

we construct a related cumulant

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\kappa\,(T_{k+1}) = \kappa\,(T_{k+1},\, T_0) = \mathrm{I}_1 - \widetilde{\mathrm{I}}_2$$

where

$$\begin{aligned}
\mathrm{I}_1 &= \int_0^\infty x^{k+1} \ln^2 x \; K(x,x)\mathrm{d}x \\
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Cumulant Structures: The Example of $\kappa_2(T)$

Now the task is to compute the difference $\delta_2(k)$

$$\begin{aligned} & \kappa\left(T_k, T\right) - \kappa\left(T_{k+1}, T_0\right) \\ = & \frac{1}{2}\int_0^\infty\int_0^\infty \left(x^k - y^k\right)(x-y)\ln x \ln y\, K(x,y)K(y,x)\mathrm{d}x\mathrm{d}y \end{aligned}$$

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that **decouples in a summation-free manner** through the Christoffel-Darboux form

$$K(x,y) \propto \sqrt{w(x)w(y)}\,\frac{L_{m-1}^{(\alpha)}(x)L_m^{(\alpha)}(y) - L_m^{(\alpha)}(x)L_{m-1}^{(\alpha)}(y)}{x-y}$$

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that **decouples in a summation-free manner** through the Christoffel-Darboux form

$$K(x,y)\propto\sqrt{w(x)w(y)}\ \frac{L_{m-1}^{(\alpha)}(x)L_m^{(\alpha)}(y)-L_m^{(\alpha)}(x)L_{m-1}^{(\alpha)}(y)}{x-y}$$

The decoupled terms are then rewritten into lower-order cumulants, which leads to the cumulant structure of $\kappa_2(T)$ as

$$\begin{aligned}\kappa(T,T)\ =\ &\kappa(R)\left(\kappa^{+}(T_0)-\kappa(T_0)\right)\left(\kappa(T_0)-\kappa^{-}(T_0)\right)-\kappa^2(R_0)\\ &+\frac{\mathrm{d}}{\mathrm{d}\alpha}\kappa(T_2)\end{aligned}$$