Third order cumulants of Products

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Cumulants

Given a non-commutative probability space (ncps) $(\mathcal{A}, \varphi)$ where $\mathcal{A}$ is an algebra and φ is a linear functional s.t. $\varphi(1) = 1$, we define the free cumulants $\kappa_n : \mathcal{A}^n \to \mathbb{C}$ by the recursive equation

$$\varphi(a_1 \cdots a_n) = \sum_{\pi \in NC(n)} \prod_{\substack{V \in \pi \\ V = \{i_1, \ldots, i_j\}}} \kappa_{|V|}(a_{i_1}, \ldots, a_{i_j}). \tag{1}$$

First few cumulants

$$
\begin{aligned}
\mathrm{E}(a) &= \kappa_1(a) = c_1(a) \\
\mathrm{E}(ab) &= \kappa_2(a,b) + \kappa_1(a)\kappa_1(b) = c_2(a,b) + c_1(a)c_1(b) \\
& \text{Thus} \\
c_2(a,b) &= \kappa_2(a,b) = \mathrm{E}(ab) - \mathrm{E}(a)\mathrm{E}(b) \\
\mathrm{E}(abc) &= \kappa_3(a,b,c) + \kappa_2(a,b)\kappa_1(c) + \kappa_2(a,c)\kappa_1(b) \\
&+ \kappa_2(b,c)\kappa_1(a) + \kappa_1(a)\kappa_1(b)\kappa_1(c) \\
c_3(a,b,c) &= \kappa_3(a,b,c) = \mathrm{E}(abc) - \mathrm{E}(c)\mathrm{E}(ab) - \mathrm{E}(b)\mathrm{E}(ac) \\
&- \mathrm{E}(a)\mathrm{E}(bc) + 2\mathrm{E}(a)\mathrm{E}(b)\mathrm{E}(c)
\end{aligned}
$$

Fourth free and classical cumulant

$$\begin{aligned} \mathrm{E}(abcd) &= \kappa_4(a,b,c,d) + \mathcal{A}(a,b,c,d) \\ &= c_4(a,b,c,d) + \mathcal{A}(a,b,c,d) + c_2(a,c)c_2(b,d). \end{aligned}$$

Here,

$$\mathcal{A}(a,b,c,d) = \sum_{\substack{\pi\in\mathcal{P}(4) \\ \pi\neq 1_4 \\ \pi\neq\{1,3\}\{2,4\}}} \kappa_\pi(a,b,c,d) = \sum_{\substack{\pi\in\mathcal{P}(4) \\ \pi\neq 1_4 \\ \pi\neq\{1,3\}\{2,4\}}} c_\pi(a,b,c,d)$$

Cumulants determine the distribution as they determine the moments (and viceversa).

Example

If $\kappa_n(a) = 0$ for any $n \neq 2$ and $\kappa_2(a) = 1$ then

$$\varphi(a^n) = |NC(n)|,$$

which determines the semicircle distribution.

Free cumulants with product as entries

Naturally, it is interesting to compute the cumulants whose entries are products; for $n_1, \dots, n_p$ integers we let

$$N_i = \sum_{j=1}^{n_i} n_j.$$

$$A_i = \prod_{j=N_{i-1}+1}^{N_i} a_j = a_{N_{i-1}+1} a_{N_{i-1}+2} \cdots a_{N_i}.$$

for $1 \leq i \leq p$ and $n_1, \dots, n_p$ integers.
Question: What are the free cumulants of

$$\kappa_p(A_1, \dots, A_p)$$

equal to in term of the cumulants of $a_i' s$.

Cumulants with product as entries formula

Theorem (Krawczyk, Speicher, 2000)

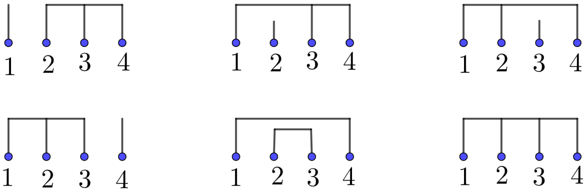
For $n = \sum_{i=1}^{p} n_i$

$$\kappa_p(A_1, \ldots, A_p) = \sum_{\substack{\pi \in NC(n) \\ \pi \vee \gamma = 1_n}} \kappa_\pi(a_1, \ldots, a_n), \tag{2}$$

where $\gamma = (1, \ldots, N_1)(N_1 + 1, \ldots, N_2) \cdots (N_{p-1} + 1, \ldots, n)$.

Example

$$\kappa_2(ab, cd) = \kappa_1(a)\kappa_3(b,c,d) + \kappa_1(b)\kappa_3(a,c,d) + \kappa_1(c)\kappa_3(a,b,d) + \kappa_1(d)\kappa_3(a,b,c) + \kappa_2(a,d)\kappa_2(b,c) + \kappa_4(a,b,c,d)$$



Second order free cumulants

Definition (Second order free cumulants)

Let $(\mathcal{A}, \varphi, \varphi^2)$ be a second order non-commutatve probability space. The *second order free cumulants* are the family of multilinear functionals $\{\kappa_{n,m} : \mathcal{A}^n \times \mathcal{A}^m \to \mathbb{C}\}_{n,m\geq 1}$ recursively defined by the following formula:

$$
\begin{aligned}
\varphi_{n,m}(a_1, \ldots, a_{n+m}) &:= \varphi^2(a_1 \cdots a_n, a_{n+1} \cdots a_{n+m}) \\
&= \sum_{(\mathcal{U},\pi)\in\mathcal{PS}_{NC}(n,m)} \kappa_{(\mathcal{U},\pi)}(a_1, \ldots, a_{n+m}) \\
&= \sum_{(\mathcal{U},\pi)\in S_{NC}(n,m)} \kappa_{(\mathcal{U},\pi)}(a_1, \ldots, a_{n+m}) \\
&+ \sum_{(\mathcal{U},\pi)\in S'_{NC}(n,m)} \kappa_{(\mathcal{U},\pi)}(a_1, \ldots, a_{n+m}).
\end{aligned}
$$

Second order cumulants of products

Mingo, Speicher and Tan, [2], showed that

$$\pi \vee \gamma_n = 1_n \Leftrightarrow \pi^{-1}\gamma_n$$

has no cycle with two elements of $N =: \{N_1, \ldots, N_p\}$. Here $\gamma_n = (1, \ldots, n)$.

Theorem (Mingo, Speicher and Tan, '09)

$$\kappa_{r,s}(A_1, \ldots, A_r, A_{r+1}, \ldots, A_{r+s}) = \sum_{(\mathcal{V},\pi)\in PS_{NC}(p,q)} \kappa_{\mathcal{V},\pi}(a_1, \ldots, a_{p+q}), \quad (3)$$

where the summation is over $(\mathcal{V}, \pi)$ such that no cycle of $\pi^{-1}\gamma_{p,q}$ has two elements of N. Here the permutation $\gamma_{p,q} = (1, \ldots, p)(p+1, \ldots, p+q) \in S_{p+q}$ with $p = n_1 + \cdots + n_r$ and $q = n_{r+1} + \cdots + n_{r+s}$.

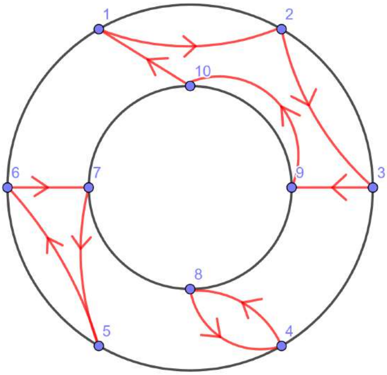
Third order cumulants with products as entries

Theorem (Arizmendi, Sigarreta and M. '25)

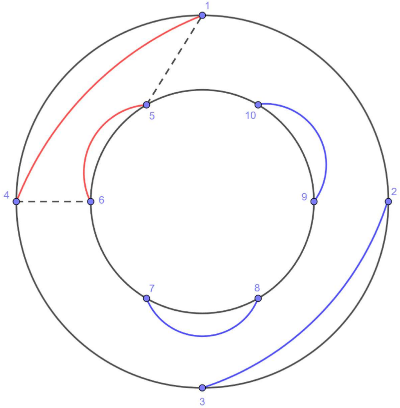
$$\kappa_{r,s,t}(A_1, \ldots, A_{r+s+t}) = \sum_{(\mathcal{V},\pi)\in\mathcal{PS}_{NC}(p,q,l)} \kappa_{(\mathcal{V},\pi)}(a_1, \ldots, a_{p+q+l}) \tag{4}$$

where the summation is over those $(\mathcal{V}, \pi) \in \mathcal{PS}_{NC}(p, q, l)$ such that $\pi^{-1}\gamma_{p,q,l}$ separates the points of $N := \{n_1, n_1 + n_2, \ldots, n_1 + \cdots + n_{r+s+t}\}$ and $p = n_1 + \cdots + n_r$, $q = n_{r+1} + \cdots + n_{r+s}$ and $l = n_{r+s+1} + \cdots + n_{r+s+t}$.

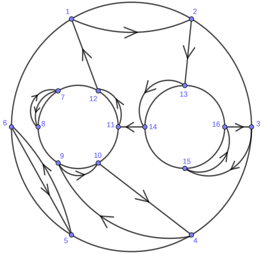
Partitioned permutations on two circles



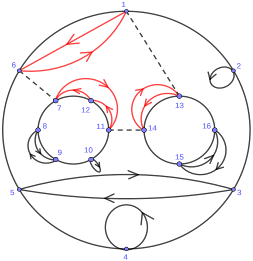
$S_{NC}(6,4)$



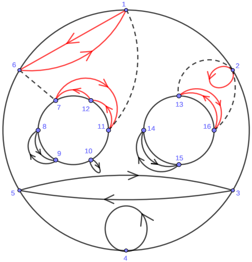
$\mathcal{PS}_{NC}(4,6)'$



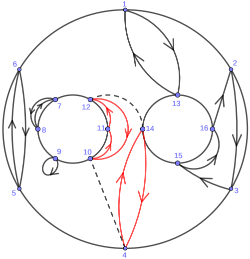
$S_{NC}(m_1, m_2)$



$\mathcal{PS}_{NC}^{(1,1,1)}(m_1, m_2, m_3)$



$\mathcal{PS}_{NC}^{(2,1,1)}(m_1, m_2, m_3)$



$\mathcal{PS}_{NC}^{(1,1)}(m_1, m_2, m_3)$

Partitioned permutations

A partitioned permutation is a pair $(\mathcal{V}, \pi)$ consisting of $\pi \in S_n$ and $\mathcal{V} \in \mathcal{P}(n)$ with $\pi \leq \mathcal{V}$. The set of partitioned permutations is denoted by $\mathcal{PS}_n$. We let,

$$|(\mathcal{V}, \pi)| = 2|\mathcal{V}| - |\pi|,$$

with $|\mathcal{V}| = n - \#(\mathcal{V})$ and $|\pi| = n - \#(\pi)$. It is satisfied,

$$|(\mathcal{V} \vee \mathcal{U}, \pi\sigma)| \leq |(\mathcal{V}, \pi)| + |(\mathcal{U}, \sigma)|.$$

For $(\mathcal{V}, \pi), (\mathcal{W}, \sigma) \in \mathcal{PS}_n$ we define their product as,

$$(\mathcal{V}, \pi) \cdot (\mathcal{W}, \sigma) = \begin{cases} (\mathcal{V} \vee \mathcal{W}, \pi\sigma) & \text{if } |(\mathcal{V} \vee \mathcal{W}, \pi\sigma)| = |(\mathcal{V}, \pi)| + |(\mathcal{W}, \sigma)|, \\ 0 & \text{otherwise} \end{cases}$$

Partitioned permutations

Definition

For $(\mathcal{U},\gamma) \in \mathcal{PS}_n$ fixed we say that $(\mathcal{V},\pi) \in \mathcal{PS}_n$ is $(\mathcal{U},\gamma)$-non crossing if,

$$(\mathcal{V},\pi) \cdot (0_{\pi^{-1}\gamma}, \pi^{-1}\gamma) = (\mathcal{U},\gamma).$$

The set of $(\mathcal{U},\gamma)$-non crossing partitioned permutations will be denote by $\mathcal{PS}_{NC}(\mathcal{U},\gamma)$.

Let $m_1, \ldots, m_r \in \mathbb{N}$ and

$$\gamma_{m_1,\ldots,m_r} := (1,\ldots,m_1)\cdots(m_1+\cdots+m_{r-1}+1,\ldots,m),$$

with $m = \sum_{i=1}^{r} m_i$. We use the notation,

$$\mathcal{PS}_{NC}(m_1,\ldots,m_r) := \mathcal{PS}_{NC}(1_m, \gamma_{m_1,\ldots,m_r}).$$

$$\mathcal{PS}_{NC}(m) = \{(0_\pi, \pi) : \pi \in NC(m)\} \cong NC(m).$$

$$\begin{aligned}\mathcal{PS}_{NC}(1_{m_1+m_2}, \gamma_{m_1,m_2}) &= \{(0_\pi, \pi) \mid \pi \in S_{NC}(m_1, m_2)\} \\ &\cup \{(\mathcal{V}, \pi) \mid \pi \in NC(m_1) \times NC(m_2), \mathcal{V} \vee \gamma = 1_n \text{ and } |\mathcal{V}| = |\pi| + 1\}\end{aligned}$$

In the first part we have $S_{NC}(m_1, m_2)$ and in the second part $S'_{NC}(m_1, m_2)$. We shall write

$$\mathcal{PS}_{NC}(m_1, m_2) = S_{NC}(m_1, m_2) \cup S'_{NC}(m_1, m_2).$$

Higher order free cumulants

Definition

Given a higher order probability space $(\mathcal{A}, (\varphi^{(s)})_{s=1}^{r})$ we let the *free cumulants of order* at most r be given by the recursive equation

$$\varphi_{(\mathcal{U},\gamma)}(a_1, \ldots, a_n) = \sum_{(\mathcal{V},\pi) \in \mathcal{PS}_{NC}(\mathcal{U},\gamma)} \kappa_{(\mathcal{V},\pi)}(a_1, \ldots, a_n).$$

higher order free cumulants for small values

$$\varphi_{1,1}(a,b) = \kappa_2(a,b) + \kappa_{1,1}(a,b),$$

$$\begin{aligned}\varphi_{1,2}(a,b,c) &= \kappa_3(a,b,c) + \kappa_3(a,c,b) + \kappa_2(a,b)\kappa_1(c) + \kappa_2(a,c)\kappa_1(b) \\ &+ \kappa_{1,1}(a,b)\kappa_1(c) + \kappa_{1,1}(a,c)\kappa_1(b) + \kappa_{1,2}(a,b,c).\end{aligned}$$

$$\begin{aligned}\varphi_{1,1,1}(a,b,c) &= \kappa_3(a,b,c) + \kappa_3(a,c,b) + \kappa_{1,2}(a,b,c) + \kappa_{1,2}(b,a,c) \\ &+ \kappa_{1,2}(c,a,b) + \kappa_{1,1,1}(a,b,c).\end{aligned}$$

Sketch of the proof

The main idea on the proof is that from the moment-cumulant formula,

$$\begin{aligned}\varphi^2(A_1 \cdots A_r, A_{r+1} \cdots A_{r+s}) &= \sum_{\pi \in S_{NC}(r,s)} \kappa_\pi(A_1, \ldots, A_{r+s}) \\ &+ \sum_{(\mathcal{V},\pi) \in \mathcal{PS}_{NC}(r,s)'} \kappa_{(\mathcal{V},\pi)}(A_1, \ldots, A_{r+s})\end{aligned}$$

and,

$$\begin{aligned}\varphi^2(A_1 \cdots A_r, A_{r+1} \cdots A_{r+s}) &= \sum_{\pi \in S_{NC}(p,q)} \kappa_\pi(a_1, \ldots, a_{p+q}) \\ &+ \sum_{(\mathcal{V},\pi) \in \mathcal{PS}_{NC}(p,q)'} \kappa_{(\mathcal{V},\pi)}(a_1, \ldots, a_{p+q})\end{aligned}$$

Sketch of the proof

Thus

$$\begin{aligned}
\kappa_{r,s}(A_1,\ldots,A_{r+s}) &= \sum_{\pi\in S_{NC}(p,q)} \kappa_\pi(a_1,\ldots,a_{p+q}) \\
&+ \sum_{(\mathcal{V},\pi)\in\mathcal{PS}_{NC}(p,q)'} \kappa_{(\mathcal{V},\pi)}(a_1,\ldots,a_{p+q}) \\
&- \sum_{\pi\in S_{NC}(r,s)} \kappa_\pi(A_1,\ldots,A_{r+s}) \\
&- \sum_{(\mathcal{V},\pi)\in\mathcal{PS}_{NC}(r,s)'\setminus(1_{r+s},\gamma_{r,s})} \kappa_{(\mathcal{V},\pi)}(A_1,\ldots,A_{r+s})
\end{aligned}$$

Possible directions

1. Higher order cases?
2. Is it possible to extend the idea of the proof on the first order case to higher orders?

THANKS

B. Krawczyk and R. Speicher. Combinatorics of Free Cumulants. *J. Combin. Theory, Ser. A.*, **90** (2000), 267-292.

J. A. Mingo, R. Speicher and E. Tan. Second order cumulants of products. *Trans. Am. Math. Soc.*, **361** (2009), 4751–4781.

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