On and around large x, N, small k expansions for log-gases and random matrices

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- A two-component log-gas
- Related two-component systems
- Structure function (spectral form factor)
- Finite size corrections

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A two-component log-gas ('82, '83)

- Boltzmann factor

$$e^{-U} = \prod_{1\le j<k\le N_1} |e^{i\phi_k} - e^{i\phi_j}| \prod_{j=1}^{N_1}\prod_{k=1}^{N_2} |e^{i\phi_j} - e^{i\theta_k}|^2 \prod_{1\le j<k\le N_2} |e^{i\theta_k} - e^{i\theta_j}|^4$$

Based on pair potential $\Phi(\theta,\theta') = -qq'\log|e^{i\theta} - e^{i\theta'}|$ $\quad q = \begin{cases} 1, & \text{species } \phi \\ 2, & \text{species } \theta \end{cases}$

- Pfaffian point process

$$\prod_{1\le j<k\le N_1} (x_k - x_j) \prod_{j_1=1}^{N_1}\prod_{j_2=1}^{N_2} (x_{j_1} - y_{j_2})^2 \times \prod_{1\le j<k\le N_2} (y_k - y_j)^4 = \det\begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 & \cdots & x_1^{N_1+2N_2-1} \\ 1 & x_2 & x_2^2 & x_2^3 & \cdots & x_2^{N_1+2N_2-1} \\ \vdots & & & & & \vdots \\ 1 & x_{N_1} & x_{N_1}^2 & x_{N_1}^3 & \cdots & x_{N_1}^{N_1+2N_2-1} \\ 1 & y_1 & y_1^2 & y_1^3 & \cdots & y_1^{N_1+2N_2-1} \\ 0 & 1 & 2y_1 & 3y_1^2 & \cdots & (N_1+2N_2-1)y_1^{N_1+2N_2-2} \\ \vdots & & & & & \vdots \\ 1 & y_{N_2} & y_{N_2}^2 & y_{N_2}^3 & \cdots & y_{N_2}^{N_1+2N_2-1} \\ 0 & 1 & 2y_{N_2} & 3y_{N_2}^2 & \cdots & (N_1+2N_2-1)y_{N_2}^{N_1+2N_2-2} \end{bmatrix}$$

- Skew inner product

$$\in \operatorname{span}\{e^{ik\phi}\}_{k=-N^*-1/2,\dots,N^*+1/2} \quad (N^* = N_1/2 + N_2)$$

$$\zeta \int_0^{2\pi} d\phi_1 \int_0^{2\pi} d\phi_2 \,\mathrm{sgn}(\phi_1 - \phi_2) f(\phi_1)\overline{g(\phi_2)} + \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 \left(f(\theta_1)\frac{d}{d\theta_2}\overline{g(\theta_2)} - g(\theta_1)\frac{d}{d\theta}\overline{f(\theta_2)} \right)$$

auxilary parameter

- Partition function

$$Z_{N_1,N_2} = \frac{1}{N_1!N_2!}\int_{[0,2\pi]^{N_1}} d\phi_1 \cdots d\phi_{N_1} \int_{[0,2\pi]^{N_2}} d\theta_1 \cdots d\theta_{N_2}\, e^{-U}$$

$$= (16\pi)^{N^*}\frac{N^*!}{(2N^*)!}[\zeta^{N_1/2}]\prod_{l=1}^{N^*}\left(\zeta + (l-1/2)^2\right)$$

Large N_1, N_2 asymptotics?

Use was made of a local CLT due to Bender '73

$$P_n(x) = x^n + a_{n-1}(n)x^{n-1} + \cdots + a_1(n)x + a_0(n) = \prod_{j=1}^{n}(x + r_j(n))$$

For $\{r_j(n)\}$ positive, normalised coefficients $a_k(n)/P_n(1)$ limit to Gaussians about a particular k^* .Need to scale ζ so that $k^* = N_1/2$. $\nu : \dfrac{N_1}{N^*} = \dfrac{\arctan\nu}{\nu}$

- Correlations

$\rho_\mathrm{b} = (N_1 + 2N_2)/L$

$$c_j(x) := \int_0^1 \frac{t^j \cos \pi\rho_b x t}{t^2 + 1/\nu^2}\, dt, \qquad s_j(x) := \int_0^1 \frac{t^j \sin \pi\rho_b x t}{t^2 + 1/\nu^2}\, dt,$$

$$\rho^T_{+1,+1}(x) = -\frac{\rho_b^2}{\nu^4}\Big((c_0(x))^2 + s_1(x)s_{-1}(x)\Big) + \frac{\pi\rho_b^2}{2\nu^2}s_1(x) \underset{x\to\infty}{\sim} -\frac{\rho_b^2}{(1+\nu^2)^2}\frac{1}{(\pi x\rho_b)^2}$$

$$\rho^T_{+1,+2}(x) = -\frac{\rho_b^2}{2\nu^2}\Big((s_1(x))^2 + c_0(x)c_2(x)\Big) \underset{x\to\infty}{\sim} -\frac{\rho_b^2\nu^2}{2(1+\nu^2)^2}\frac{1}{(\pi x\rho_b)^2}$$

$$\rho^T_{+2,+2}(x) = -\frac{\rho_b^2}{4}\Big((c_2(x))^2 + s_1(x)s_3(x)\Big) \underset{x\to\infty}{\sim} -\frac{\rho_b^2\nu^4}{4(1+\nu^2)^2}\frac{1}{(\pi x\rho_b)^2}$$

- Sum rules: Charge-charge correlation

$$c_{(1)}(\vec r) := \sum_{j=1}^{N} q_j \delta(\vec r - \vec r_j)$$

$$C_{(2)}(\vec r, \vec s) = \langle c_{(1)}(\vec r) c_{(1)}(\vec s)\rangle - \langle c_{(1)}(\vec r)\rangle\langle c_{(1)}(\vec s)\rangle,$$

$$C_{(2)}(x,0) = \rho^T_{+1,+1}(x,0) + 4\rho^T_{+1,+2}(x,0) + 4\rho^T_{+2,+2}(x,0) + \delta(x)\Big(\rho_{+1}(x) + 4\rho_{+2}(x)\Big).$$

$$\int_{-\infty}^{\infty} C_{(2)}(x,0)\,dx = 0.$$ (perfect screening)

$\beta = 1$

$$C_{(2)}(x,x') \underset{|x-x'|\to\infty}{\sim} -\frac{1}{\beta\pi^2(x-x')^2} \quad \Leftrightarrow \hat C(k) \underset{k\to 0}{\sim} \frac{|k|}{\pi\beta}$$

Related studies:

★ Generalised plasma ('84 with B. Jancovici, '11 with C. Sinclair)

$$\prod_{1\le j<k\le N_1} |e^{i\phi_k} - e^{i\phi_j}|^2 \prod_{j=1}^{N_1}\prod_{k=1}^{N_2} |e^{i\phi_j} - e^{i\theta_k}|^2 \prod_{1\le j<k\le N_2} |e^{i\theta_k} - e^{i\theta_j}|^4$$

Changed

- Partitian function can be evaluated as a product of gamma functions for general exponents $(g, g, g+2)$. Follows from the theory of Jack polynomials with prescribed symmetry ('96 with T. Baker)

- Pfaffian point process $\displaystyle\prod_{1\le j<k\le N} (z_k - z_j) = \mathrm{Pf}\left[\frac{(z_k^{N/2} - z_j^{N/2})^2}{z_k - z_j}\right]$

- Skew inner product $\zeta a_{j,k} + b_{j,k}$

$$\int_0^{2\pi} z^{-(N_1+2N_2-2)}\left(p_{j-1}(z)p'_{k-1}(z) - p'_{j-1}(z)p_{k-1}(z)\right) d\theta$$

$$\in \operatorname{span}\{1, z, \dots, z^{N_1+2N_2-1}\}$$

$$\int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 \, z_1^{-N_2} z_2^{-N_2} \frac{(z_2^{-N_1/2} - z_1^{-N_1/2})}{z_2^{-1} - z_1^{-1}} p_{j-1}(z_1)p_{j-1}(z_2)$$

- Correlations e.g.

$$-\rho_a^2 e^{-\pi i \rho_a (x+y)} \int_{-1/2}^{1/2} t e^{2\pi i \rho_a (x-y)t} \, dt$$

$$\rho_{aa}^T(x,0) = -\left(S_{aa}(x,0)S_{aa}(0,x) + D_{aa}(x,0)I_{aa}(x,0)\right)$$

$$\rho_a \int_0^1 e^{2\pi i \rho_a (x-y)t} \, dt$$

$$\rho_b e^{\pi i \rho_a (x+y)} \int_0^1 \frac{1}{\rho_b t + \rho_a/2} \left(e^{-2\pi i(\rho_b t + \rho_a/2)(x-y)} - \text{c.c.}\right) dt$$

- Sum rules

$$\int_{-\infty}^{\infty} \rho_{aa}^T(x,0)\, dx = -\rho_a \qquad \int_{-\infty}^{\infty} \rho_{ab}^T(x,0)\, dx = 0 \qquad \int_{-\infty}^{\infty} \rho_{bb}^T(x,0)\, dx = -\rho_b$$

$$\rho_{aa}^T(x,0) \underset{x\to\infty}{\sim} -\frac{g_{aa}}{\pi^2 \Delta x^2} \qquad \rho_{ab}^T(x,0) \underset{x\to\infty}{\sim} \frac{g_{ab}}{\pi^2 \Delta x^2} \qquad \rho_{bb}^T(x,0) \underset{x\to\infty}{\sim} -\frac{g_{bb}}{\pi^2 \Delta x^2}$$

$$g_{aa}g_{bb} - g_{ab}^2$$

★ GinOE (real Ginibre ensemble)

PDF for sector with k real eigenvalues (Lehmann&Sommers, Edelman)

$$C_N^{\rm g} \frac{2^{(N-k)/2}}{k!((N-k)/2)!} \prod_{s=1}^{k} (\omega^{\rm g}(\lambda_s))^{1/2} \prod_{j=1}^{(N-k)/2} \omega^{\rm g}(z_j) \left| \Delta(\{\lambda_l\}_{l=1,\dots,k} \cup \{x_j \pm i y_j\}_{j=1,\dots,(N-k)/2}) \right|$$

$$e^{-|z|^2} e^{2y^2} \mathrm{erfc}(\sqrt{2}y)$$

Vandermonde product

- Prior knowledge: A two-component Pfaffian point process (Akemann&Kanzieper, Sinclair)
- Skew polynomial formalism introduced in '07 with Nagao

$$\zeta \alpha_{j,k} + \beta_{j,k}$$

(only require $\zeta = 1$ for correlations)

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \; (\omega^{\rm g}(x)\omega^{\rm g}(y))^{1/2} p_{j-1}(x) p_{k-1}(y) \mathrm{sgn}\,(y-x)$$

$$2i \int_{\mathbb{R}^2_+} dxdy \; v(x,y) \omega^{\rm g}(z) \Big(p_{j-1}(x+iy) p_{k-1}(x-iy) - p_{k-1}(x+iy) p_{j-1}(x-iy) \Big)$$

- Skew orthogonal polynomials

$$p^{\mathrm{g}}_{2n}(z) = z^{2n}, \qquad p^{\mathrm{g}}_{2n+1}(z) = z^{2n+1} - 2nz^{2n-1} = -\, e^{z^2/2}\frac{d}{dz}e^{-z^2/2}p_{2n}(z)$$

(GOE structure)

Found using computer algebra, then verified. Much better

Akemann, Kieburg and Philips '09

$$p_{2n}(z) = \langle \det(z\mathbb{I}_{2n} - G)\rangle, \quad p_{2n+1}(z) = zp_{2n}(z) + \langle \det(z\mathbb{I}_{2n} - G)\mathrm{Tr}\, G\rangle$$

$$= z^{2n} \qquad\qquad = z^{2n+1} - \langle \mathrm{Tr}\, G^2\rangle z^{2n-1}$$

$$g_{jk} \overset{\mathrm{d}}{=} -\, g_{jk}$$

- Limiting correlation kernel e.g. for real eigenvalues

$$K^{\mathrm{r,b}}_{\infty}(x,y) = \begin{bmatrix} \frac{1}{2\sqrt{2\pi}}(y-x)e^{-(x-y)^2/2} & \frac{1}{\sqrt{2\pi}}e^{-(x-y)^2/2} \\ -\frac{1}{\sqrt{2\pi}}e^{-(x-y)^2/2} & \mathrm{sgn}(x-y)\mathrm{erfc}(|x-y|/\sqrt{2}) \end{bmatrix}.$$

- Sum rules

Real eigenvalue at the origin

$$2\int_{\mathbb{C}_+} \rho^{\mathrm{c,b},T}_{(2),\infty}(0,z)\, d^2 z + \int_{-\infty}^{\infty} \rho^{\mathrm{r,b},T}_{(2),\infty}(0,y)\, dy = -\rho^{\mathrm{r}}\,.$$

Complex eigenvalue at z_0

$$2\int_{\mathbb{C}_+} \rho^{\mathrm{c,b},T}_{(2),\infty}(z_0, z)\, d^2 z + \int_{-\infty}^{\infty} \rho^{(\mathrm{c,r}),\mathrm{b},T}_{(2),\infty}(z_0, x)\, dx = -2\rho^{\mathrm{c,b}}_{(1),\infty}(z_0)$$

Linear in s

$\Leftrightarrow$ Compressible gas

- Gap probability

$$E^{\mathrm{r,b}}(0;(0,s);\xi) \underset{s\to\infty}{\sim} e^{-c(\xi)s+O(1)}, \quad c(\xi) = -\frac{1}{4\pi}\int_{-\infty}^{\infty} \log\left(1-(2\xi-\xi^2)e^{-u^2/2}\right) du$$

$\xi = 1$

$$= \frac{1}{2\sqrt{2\pi}}\zeta(3/2)$$

Relates to annihilation process $A + A \to \emptyset$ and to $p^{\mathrm{r}}_{N,0}$

Probability of no real eigenvalues

(Kanzieper et al '16)

★ Local CLTs (with J. Lebowitz '14)

Prob. k eigenvalues in J

Consider $N(J)$ $\quad \Pr\Big(N(J) \leq y\Big) = \sum_{k=0}^{[y]} E(k;J)$.

number of eigenvalues in J

- Costin Lebowitz '95 : for self adjoint DPP with $\sigma_J \underset{|J|\to\infty}{\to} \infty$

$$\lim_{|J|\to\infty} \frac{(N(J)-\mu_J)}{\sigma_J} \overset{\mathrm{d}}{=} \mathrm{N}[0,1], \quad \text{i.e.} \quad \lim_{|J|\to\infty} \sup_{x\in(-\infty,\infty)} \left| \sum_{k\leq\sigma_J x+\mu_J} E(k;J) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt \right| = 0$$

Possibly ∞

real, positive

- Following Bender '73, if $\sum_{k=0}^{N} z^k E(k;J) = A_N \prod_{j=0}^{N} (z + r_j(N))$

Fredholm det for DPP

then stronger local CLT

$$\lim_{|J|\to\infty} \sup_{x\in(-\infty,\infty)} \left| \sigma_J E([\sigma_J x + \mu_J];J) - \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \right| = 0.$$

(i.e. there is a regime where scaled individual $E(k;J)$ are Gaussians).

★ Two-component log-gas line version (with S.-H. Li ’21)

Boltzmann factor

$$\prod_{1\le j<k\le N_1} |\alpha_k - \alpha_j| \prod_{1\le j<k\le N_2} |\beta_k - \beta_j|^4 \prod_{j=1}^{N_1}\prod_{k=1}^{N_2} |\alpha_j - \beta_k|^2 e^{-V(\alpha_j)-2V(\beta_k)}$$

Gaussian case

$V(x) = x^2/2$

Rider et al ‘13

- Partition function

For all classical weights Gaussian, Laguerre, Jacobi, Cauchy,

$$Z_{N_1,N_2} \propto [\zeta^{N_1/2}]P_{N^*}(\zeta)$$

hypergeometric polynomial

$$\begin{cases} {}_1F_1, & \text{Laguerre polynomial (G)} \\ {}_3F_2, & \text{continuous Hahn polynomial (L)} \\ {}_4F_3, & \text{Wilson polynomial (J, C)} \end{cases}$$

- Skew orthogonal polynomials $\sum_{l=0}^{k} c_k p_k(x)$

classical polynomial

★ Structure function (spectral form factor)

- Defined as the FT of the density-density (or charge-charge) correlation
- Consider the particular case of the bulk scaled ($\rho_{\mathrm{b}} = 1$) circular β ensemble

$$S(k;\beta) = \int_{-\infty}^{\infty} \left(\rho_{(2)}^{\mathrm{T}}(x;\beta) + \delta(x) \right) e^{ikx}\, dx$$

e.g. for $\beta = 2$

$$\rho_{(2)}^{\mathrm{T}}(x;\beta) = -\left(\frac{\sin \pi x}{\pi x} \right)^2 \qquad S(k;\beta) = \begin{cases} |k|/2\pi, & |k| < 2\pi \\ 1, & |k| \geq 2\pi \end{cases}$$

Perfect screening, general $\beta > 0$, $\implies S(k;\beta)|_{k=0} = 0.$

Linear response, screening an external charge in long wavelength limit

$$\implies S(k;\beta) \underset{k\to 0}{\sim} |k|/\beta\pi$$

Makes use of $\mathrm{FT}(-\log|x|) = \dfrac{\pi}{|k|}$

pair potential

- At next order, hydrodynamical argument involving the pressure gives

$$S(k;\beta) \sim \frac{|k|}{\pi\beta} - \Big(1 - \frac{\beta}{2}\Big)\Big(\frac{k}{\pi\beta}\Big)^2$$

$A = \sum a(x_j)$ Linear statistic

- Aside: An immediate consequence. Start with $\mathrm{Var}\, A = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{a(k)}|^2 S(k;\beta)\, dk$

Let $a(x) \mapsto a(x/L), L$ large

$$\mathrm{Var}\, A = \frac{1}{2\pi^2\beta}\int_{-\infty}^{\infty} |\hat{c}(k)|^2 |k|\, dk + \frac{1}{L}\frac{1}{2\pi(\pi\beta)^2}(1-\beta/2)\int_{-\infty}^{\infty} |\hat{c}(k)|^2 k^2\, dk + \cdots$$

Expansion in $1/L$

- Suppose $k > 0,\ 0 < k < \min(2\pi, \beta\pi)$.

Functional equation $\frac{\beta\pi}{k}S(k;\beta) = \frac{4\pi}{\beta k}S(k;4/\beta)\Big|_{k\mapsto -2k/\beta}$

Degree j poly.

Consistent with $\frac{\beta\pi}{k}S(k;\beta) = 1 + \sum_{j=1}^{\infty} p_j(\beta/2)(k/\pi\beta)^j$

$p_j(1/x) = (-x)^{-j}p_j(x)$

('00 with B. Jancovici and D. McAnally)

- For the 2d Coulomb gas it was known

$$S(\vec{k};\beta) \sim \frac{|\vec{k}|^2}{2\pi\beta} - \left(1 - \frac{\beta}{4}\right)\left(\frac{|\vec{k}|^2}{2\pi\beta}\right)^2 + \left(\frac{\beta}{4} - \frac{3}{2}\right)\left(\frac{\beta}{4} - \frac{2}{3}\right)\left(\frac{|\vec{k}^2|}{2\pi\beta}\right)^3$$

Due to Kalinay et al '00
Later Wiegmann et al

However, thought to break down at higher order in $|\vec{k}|^2$

- For the circular β ensemble use a loop equation formalism applied to (with N. Witte '15, B.-J. Shen '23)

$$\prod_{l=1}^{N-1} |1 - e^{i\theta}|^{\beta} \prod_{1 \le j < k \le N-1} |e^{i\theta_k} - e^{i\theta_j}|^{\beta}$$

conditioned eigenvalue is at $\theta = 0$

Compute $\left\langle G(x) \right\rangle, \quad G(x) = \sum_{j=1}^{N} \frac{1}{x - e^{i\theta}}$

series in orders of 1/N

generating function for Fourier coefficients of

$$\frac{1}{2\pi}\rho_{(2),N}(\theta;\beta) = \rho_{(1),N-1}(\theta;\beta)$$
$$= (N-1) + \sum_{j \neq 0} c_j(N) e^{ij\theta}$$

- To compute $1/N$ expansion of $\left\langle G(x) \right\rangle$ requires $1/N$ expansion of $\left\langle G(x_1)G(x_1) \right\rangle^T, \dots$ (triangular structure) . Also require $c_j(N) \mapsto (2\pi/N)c_{Nj}(N)$ to account for bulk scaling.

- Result:
$$\frac{\beta\pi}{k} S(k;\beta) = 1 + \sum_{j=1}^{\infty} p_j(\beta/2)(k/\pi\beta)^j$$
('00 with B. Jancovici and D. McAnally)

$$p_1(x) = 1 - x, \quad p_2(x) = 1 - \frac{11x}{6} + x^2, \quad p_3(x) = (1-x)\left(1 - \frac{3x}{2} + x^2\right), \dots$$

Up to $p_{10}(x)$

(Aside: Each $p_j(x)$ has all zeros on the unit circle, which interlace)

- Question (with B.-J. Shen '25) . Small $|k|$, large N expansion of $S_N(k;\beta)$?

$$S_N(k;\beta) = \int_0^{2\pi} e^{ik\theta} \rho^T_{(2),N}(\theta;\beta)\, d\theta + \frac{N}{2\pi},$$

Bulk scaling.

$$\tilde{S}(\tau;\beta) := \frac{2\pi}{N} S_N(\tau N;\beta)$$

- $\beta = 2$

$$\tilde{S}_N(\tau;\beta)\Big|_{\beta=2} = \begin{cases} |\tau|, & |\tau| < 1 \\ 1, & |\tau| \geq 1, \end{cases}$$

No dependence on N

- $\beta = 1,4$ Exact expressions in terms of digamma (Haake et al '96)

$$\Longrightarrow \quad \tilde{S}_N(\tau;\beta) \sim \tilde{S}_{0,\infty}(\tau;\beta) + \frac{1}{N^2}\tilde{S}_{1,\infty}(\tau;\beta) + \frac{1}{N^4}\tilde{S}_{2,\infty}(\tau;\beta) + \cdots$$

$1/N^2$ expansion

e.g. $\beta = 1$

$$\tilde{S}_N(\tau;\beta)\Big|_{\beta=1} = \begin{cases} 2|\tau| - |\tau|\log(1+2|\tau|) - \dfrac{|\tau|}{6N^2}\left(1 - \dfrac{1}{(1+2|\tau|)^2}\right) + \cdots & |\tau| \leq 1 \\ 2 - |\tau|\log\dfrac{2|\tau|+1}{2|\tau|-1} + \dfrac{|\tau|}{N^2}\dfrac{4|\tau|}{3(1-(2|\tau|)^2)^2} + \cdots & |\tau| \geq 1, \end{cases}$$

Observation

$$\tilde{S}_{1,\infty}(\tau;\beta) = c_\beta \tau^2 \frac{d^2}{d\tau^2}\tilde{S}_{0,\infty}(\tau;\beta),$$

known

$$\tilde{S}_{2,\infty}(\tau;\beta) = d_\beta\left(\tau^4\frac{d^4}{d\tau^4}\tilde{S}_{0,\infty}(\tau;\beta) + 8\tau^3\frac{d^3}{d\tau^3}\tilde{S}_{0,\infty}(\tau;\beta) + 12\tau^2\frac{d^2}{d\tau^2}\tilde{S}_{0,\infty}(\tau;\beta)\right)$$

- Loop equation formalism allows for the computation of series expansions of $\tilde{S}_{0,\infty}(\tau;\beta), \tilde{S}_{1,\infty}(\tau;\beta), \tilde{S}_{2,\infty}(\tau;\beta)$ in τ for all $\beta > 0$. Consistency with the DEs is found.

(albeit to low order)

- Recalling

$$S_N(k;\beta) = \int_0^{2\pi} e^{ik\theta}\rho^T_{(2),N}(\theta;\beta)\,d\theta + \frac{N}{2\pi},$$

what does this say about the large N expansion of $(2\pi/N)^2\rho_{(2),N}(2\pi x/N;\beta)$?.

bulk scaled two-point function.

e.g. $\beta = 2$

$1/N^2$ expansion

$$\left(\frac{2\pi}{N}\right)^2\rho^{\mathrm{CUE}}_{(2)}\left(\frac{2\pi X}{N},\frac{2\pi Y}{N}\right) = 1 - \left(\frac{\sin\pi(X-Y)}{\pi(X-Y)}\right)^2 - \frac{1}{3N^2}\sin^2\pi(X-Y) + \mathrm{O}\left(\frac{1}{N^4}\right),$$

meaning of FT?.

- Recall

$$\tilde{S}_{1,\infty}(\tau;\beta) = c_\beta \tau^2 \frac{d^2}{d\tau^2}\tilde{S}_{0,\infty}(\tau;\beta),$$

formal

$$\int_{-\infty}^{\infty}\left(\rho^{\mathrm{T}}_{(2),1,\infty}(x;\beta)+\delta(x)\right)e^{i\tau x}\,dx$$

exact

$$\int_{-\infty}^{\infty}\left(\rho^{\mathrm{T}}_{(2),0,\infty}(x;\beta)+\delta(x)\right)e^{i\tau x}\,dx$$

- Suggests

$$\rho_{(2),1,\infty}(s,0) = c_\beta \frac{d^2}{ds^2}\left(s^2\rho_{(2),0,\infty}(s;\beta)\right)$$

$1/N^2$ term

bulk scaled limit

- Can be verified for all even β and $\beta = 1$.

- Breaks down at next order

$$\rho_{(2),2,\infty}(x;\beta) \neq d_\beta\left(\frac{d^4}{dx^4} x^4 \rho_{(2),0,\infty}(x;\beta) + 8\frac{d^3}{dx^3} x^3 \rho_{(2),0,\infty}(x;\beta) + 12\frac{d^2}{dx^2} x^2 \rho_{(2),0,\infty}(x;\beta) \right)$$

order $1/N^4$ term

bulk scaled limit

- Corrections for spacings?

conditioned gaps

$$\mathscr{E}_N^{(\cdot)}((0,\phi);\xi) := \sum_{k=0}^{N} (1-\xi)^k E_N^{(\cdot)}(k;(0,\phi)) = 1 + \sum_{k=1}^{N} \frac{(-\xi)^k}{k!} \int_0^\phi d\theta_1 \cdots \int_0^\phi d\theta_k\, \rho_{(k)}^{(\cdot)}(\theta_1,\ldots,\theta_k),$$

$$\mathscr{P}_N^{(\cdot)}(x;\xi) := \sum_{k=0}^{N-2} (1-\xi)^k p_N^{(\cdot)}(k;x) = \rho_{(2)}^{(\cdot)}(0,x) + \sum_{k=1}^{N-2} \frac{(-\xi)^k}{k!} \int_0^x dx_1 \cdots \int_0^x dx_k\, \rho_{(k+2)}^{(\cdot)}(0,x,x_1,\ldots,x_k)$$

conditioned spacings

- Can establish that

$$\mathscr{P}^{\mathrm{bulk}}_{1,\beta}(s;\xi) = -\frac{1}{6\beta}\frac{d^2}{ds^2}\left(s^2\mathscr{P}^{\mathrm{bulk}}_{0,\beta}(s;\xi)\right),$$

$1/N^2$ term

bulk scaled limit

valid for $\beta = 1,2,4$, with the asymptotic expansion in powers of $1/N^2$ (conjectured for all $\beta > 0$)

- Taking $\xi \to 0$ reclaims the result for the 2-point function.

- Setting $\xi = 1$ corresponds to the usual gap probability.

e.g. $\beta = 2$ $\quad P^{\mathrm{bulk}}_{0,\beta=2}(s) = \dfrac{d^2}{ds^2}\det\left(\mathbb{I} - \mathbb{K}_s\right) \approx \dfrac{32s^2}{\pi^2}e^{-4s^2/\pi}$

Wigner surmise

$p^{\mathrm{W}}_{\beta=2}(s)$

integral operator on $(0,s)$ with sine kernel

- From ('15 with A. Mays, and '17 with F. Borneman and A. Mays)

$$p^{\mathrm{bulk}}_{1,\beta=2}(s) = -\frac{d^2}{ds^2}\det(\mathbb{I} - \xi\mathbb{K}_s)\mathrm{Tr}((\mathbb{I} - \xi\mathbb{K}_s)^{-1}\xi\mathbb{L}_s)\Big|_{\xi=1}.$$

kernel

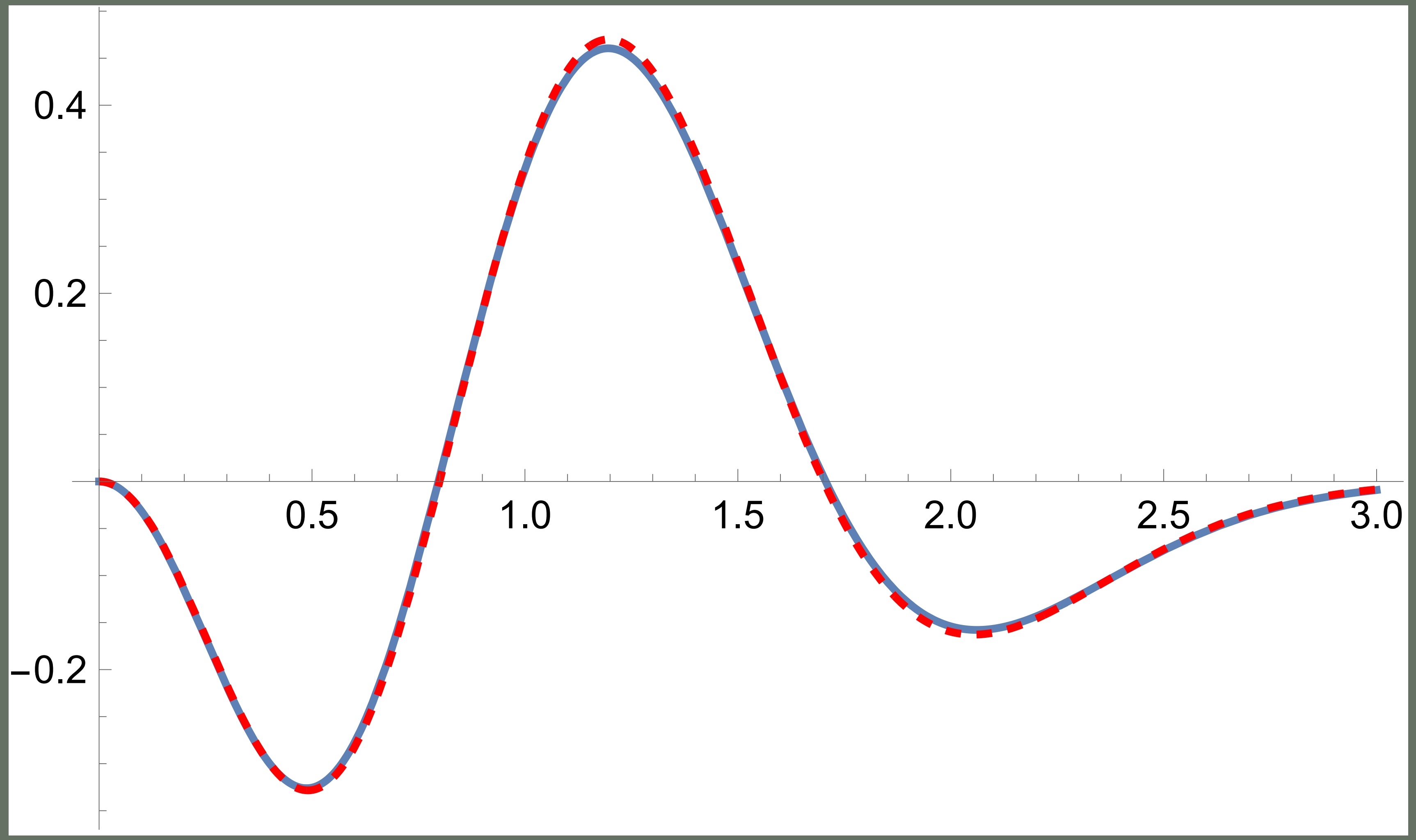
$(\pi(x-y)/6)\sin(\pi(x-y))$.

using identity

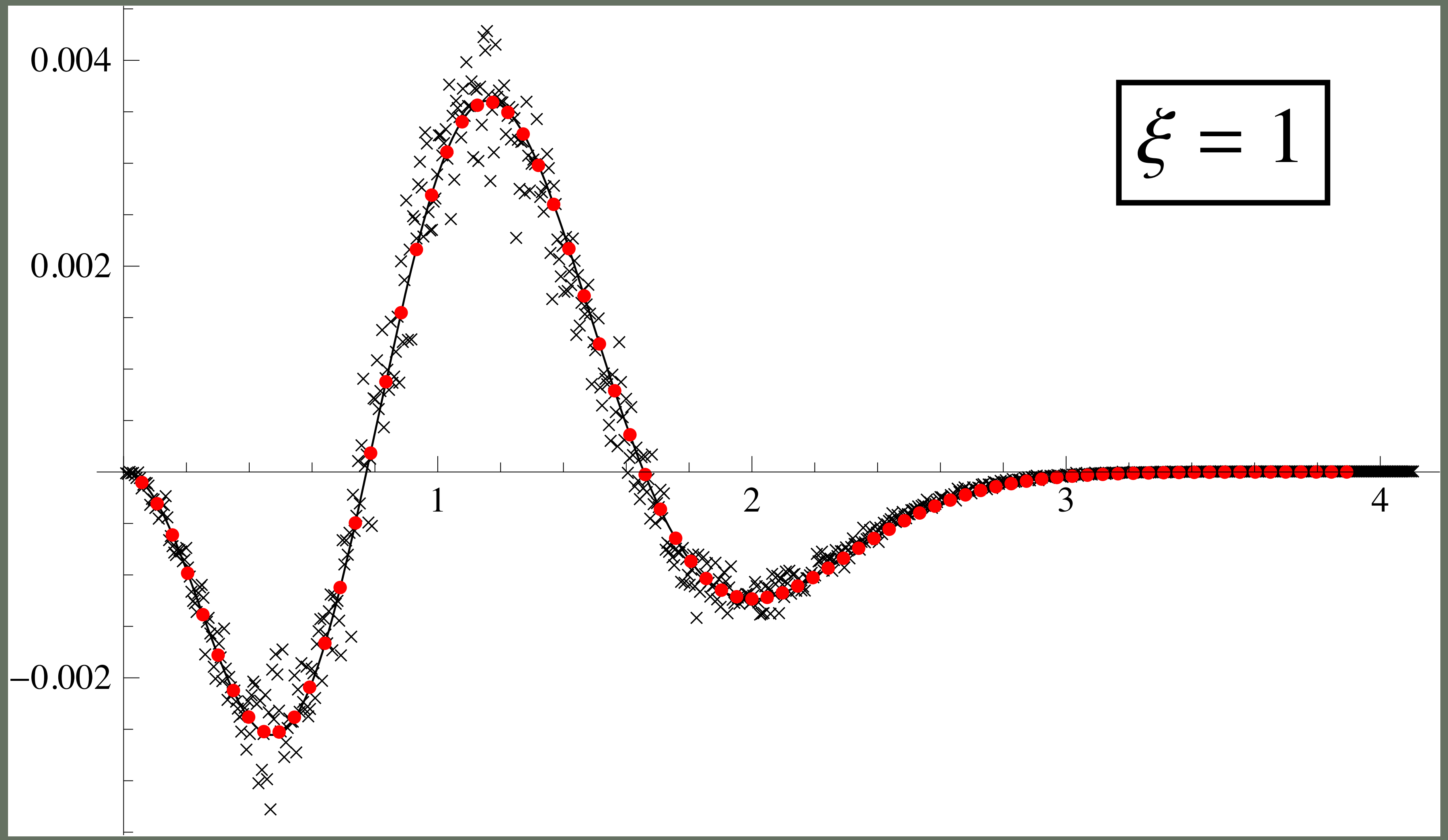
and Wigner surmise

$$\approx -\frac{1}{12}\frac{d^2}{ds^2}\left(s^2 p^{\mathrm{W}}_{\beta=2}(s)\right)$$

cf. Odlyzko's Riemann zeros data (10^9 zeros at $\approx$ height 10^{23})
(from '15 with A. Mays, following Keating-Snaith, Bogomolny et al.)



Graphics from '25 with B-J. Shen



Thank you all!

Log-gas/ RMT/ Selberg integral communities

Research fellows/ graduate students

Collaborators

Mentors/ supporters

Family

- Application