On the moments of the derivative of CUE characteristic polynomials inside the unit disc

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Contents

1. Some background on connections between number theory and random matrix theory

Riemann zeta function

When Re(s)>1,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p (1 - \frac{1}{p^s})^{-1}.$$

Pole: $s = 1$

Functional equation:

$$\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s) = \pi^{-(1-s)/2}\Gamma(\frac{1-s}{2})\zeta(1-s)$$

Trivial zeros: $s = -2, -4, -6, -8, \cdots$

Non-trivial zeros: $0 < \mathrm{Re}(s_n) < 1$.

Riemann Hypothesis: $\forall n$, $\mathrm{Re}(s_n) = 1/2$, i.e., $s_n = \frac{1}{2} + \imath t_n$ for $t_n \in \mathbb{R}$.

Connections with random matrix theory

Set $w_n = \frac{t_n}{2\pi} \log \frac{|t_n|}{2\pi}$.

Conjecture (Montgomery, 1972)

Let $f(x)$ be an integrable function with a compact support,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n\neq m\leq N} f(w_n - w_m) = \int_{-\infty}^{\infty} f(x)\left(1 - \left(\frac{\sin(\pi x)}{\pi x}\right)^2 + \delta(x)\right) dx,$$

where $\delta(x)$ is the Dirac delta function such that $\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0)$.

Montgomery's conjecture can be proved for a restricted class of test functions $f(x)$, e.g., if the Fourier transform of $f(x)$ is supported on $[-1, 1]$.

Denote the eigenvalues of an $N \times N$ unitary matrix A by $e^{i\theta_n(A)}$ and set $\phi_n(A) = \frac{N}{2\pi}\theta_n(A)$.

Theorem (CUE Pair Correlation - Dyson, 1963)

$$\lim_{N\to\infty}\int_{U(N)} \frac{1}{N}\sum_{n\neq m\leq N} f(\phi_n(A)-\phi_m(A))dA_N$$
$$= \int_{-\infty}^{\infty} f(x)\left(1-\left(\frac{\sin(\pi x)}{\pi x}\right)^2+\delta(x)\right)dx.$$

The n-point correlation function

$$R_n(Q) = \frac{1}{N}\#\{j_1,\ldots,j_n \leq N \text{ distinct } : (w_{j_1}-w_{j_2},\cdots,w_{j_{n-1}}-w_{j_n})\in Q\}$$

for a box $Q \subset \mathbb{R}^{n-1}$.

Summarize the above,

Let $A \in U(N)$ be taken from the Circular Unitary Ensemble (CUE) of random matrices.

The characteristic polynomial of A:

$$\Lambda_N(z) := \det(I - zA^*) = \prod_{n=1}^{N} (1 - ze^{-\mathrm{i}\theta_n}),$$

where $e^{\mathrm{i}\theta_1}, \ldots, e^{\mathrm{i}\theta_N}$ are the eigenvalues of A.

$$\mathbb{E}[|\Lambda_N(z)|^s] := \int_{U(N)} |\Lambda_N(z)|^s d\nu_{Haar}, \mathbb{E}[|\Lambda_N'(z)|^s] := \int_{U(N)} |\Lambda_N'(z)|^s d\nu_{Haar}$$

The distribution of zeros of Λ_N $\xrightarrow{\text{model}}$ The distribution of zeros of ζ

The distribution of zeros of Λ_N' $\xrightarrow{\text{model}}$ The distribution of zeros of ζ'

A.Speiser (1934):
RH $\Leftrightarrow$ $\zeta'(s)$ has no nonreal zeros in the region $\{\sigma + \mathrm{i}t : \sigma < 1/2\}$.

Soundararajan's Conjecture (1998) (horizontal distribution):

$$\frac{\#\{\sigma+it:\sigma\le\frac{1}{2}+\frac{c}{\log T},0\le t\le T,\zeta'(\sigma+\mathrm{i}t)=0\}}{\#\{\sigma+it:0\le t\le T,\zeta'(\sigma+\mathrm{i}t)=0\}}$$

$$\xrightarrow{T\to\infty}\quad \rho(c)=\begin{cases}0 & \text{if } c\le 0\\ \in(0,1) & \text{if } c>0\\ \to 1 & \text{if } c\to\infty\end{cases}$$

Correspondingly (radial distribution): $\log T$ above correspond to N below,

$$\frac{\mathbb{E}[\#\{z:\Lambda_N'(z)=0,1\ge|z|\ge 1-\frac{c}{N}\}]}{N}\quad\xrightarrow{N\to\infty}\quad\rho(c).$$

A question proposed by Brian Conrey

Jessen's formula:

$$\frac{1}{2\pi}\int_0^{2\pi}\log|\Lambda_N'(re^{i\theta})|d\theta - \log|\Lambda_N'(0)| = \int_0^r \frac{n_N(t)}{t}dt,$$

where $n_N(r)$ be the number of zeros of $\Lambda_N'(z)$ inside the disc of radius r centered at the origin.

By the translation-invariance of the Haar measure

$$\mathbb{E}[\log|\Lambda_N'(re^{i\theta})|] \qquad \text{(independent of } \theta\text{)}$$

we have

$$\begin{aligned}\int_0^r \frac{\mathbb{E}[n_N(t)]}{t}dt &= \mathbb{E}[\log|\Lambda_N'(r)|] - \mathbb{E}[\log|\Lambda_N'(0)|] \\ &= \frac{d}{ds}\mathbb{E}[|\Lambda_N'(r)|^s]\Big|_{s=0} - \frac{d}{ds}\mathbb{E}[|\Lambda_N'(0)|^s]\Big|_{s=0}.\end{aligned}$$

So this is one motivation to study

$$\mathbb{E}[|\Lambda_N'(r)|^s].$$

Another background: A connection to the moments of the Riemann zeta function on the critical line

Keating-Snaith's work and conjecture (2000):

$$g_s := \lim_{N\to\infty} \frac{1}{N^{s^2}} \mathbb{E}[|\Lambda_N(1)|^{2s}] = \frac{G^2(s+1)}{G(2s+1)} \quad \text{for } \mathrm{Re}(s) > -1/2,$$

where $G(s)$ is the Barnes G-function.

Keating-Snaith's Conjecture

$$\frac{1}{T}\int_0^T |\zeta(\frac{1}{2}+\mathrm{i}t)|^{2s} dt \sim a_s g_s (\log T)^{s^2}, \quad T \to \infty.$$

In number theory, $g_1 = 1, g_2 = 2$ and conjectured $g_3 = 42, g_4 = 24024$. This coincides with $\frac{G^2(s+1)}{G(2s+1)}$ with $s = 1, 2, 3, 4$.

More evidence was shown in the work on a “hybrid" representation of the Riemann zeta function by Gonek, Hughes and Keating (2007).

Here a_s is the arithmetic factor

$$a_s = \prod_p \left(1 - \frac{1}{p}\right)^{s^2} \sum_{m=0}^{\infty} \left(\frac{\Gamma(m+s)}{m! \cdot \Gamma(s)}\right)^2 p^{-m}.$$

Hughes' conjecture (2001):

$$\frac{1}{T}\int_0^T |\zeta'(\frac{1}{2}+\mathrm{i}t)|^{2s} dt \sim a_s b_s (\log T)^{s^2+2s}, \quad T \to \infty$$

$$\text{where } b_s := \lim_{N\to\infty} \frac{1}{N^{s^2+2s}} \mathbb{E}[|\Lambda_N'(1)|^{2s}]$$

- Hughes (2001): derived an expression for b_s
- Conrey, Rubinstein, and Snaith (2006): gave an expression for b_s in terms of a Hankel determinant
- Forrester and Witte (2006): established a connection of b_s to a solution of σ-Painlevé III$'$
- Assiotis, Keating and Warren (2022): proved the existence of b_s for noninteger and real s, and gave a probabilistic representation of b_s
- ......

Question 1: What is about

$$\mathbb{E}[|\Lambda_N(z)|^{2s}] \text{ and } \mathbb{E}[|\Lambda_N'(z)|^{2s}]$$

when $|z| < 1$?

Question 2: Is there a connection between them and Number theory, specifically

$$\lim_{T\to\infty} \frac{1}{T}\int_1^T |\zeta'(\sigma + \mathrm{i}t)|^{2s} dt \quad \text{for } \sigma > 1/2?$$

Forrester and Keating (2004): for $|z| < 1$,

$$\lim_{N\to\infty} \mathbb{E}[|\Lambda_N(z)|^{2s}] = \frac{1}{(1-|z|^2)^{s^2}}.$$

Our focus is on the study of

$$\mathbb{E}[|\Lambda_N'(z)|^{2s}].$$

2. Main results

Global regime

Theorem 1

For any fixed z with $|z| < 1$ and any $s \in \mathbb{C}$ with $\mathrm{Re}(s) > -1$, we have

$$\lim_{N\to\infty} \mathbb{E}\left(|\Lambda_N'(z)|^{2s}\right) = \frac{e^{-s^2|z|^2}\Gamma(s+1)}{(1-|z|^2)^{s^2+2s}}\, {}_1F_1(s+1,1;s^2|z|^2)$$

where ${}_1F_1(a,b;z)$ is the confluent hypergeometric function of the first kind given by

$${}_1F_1(a,b;z) = \sum_{k=0}^{\infty} \frac{a^{(k)}}{b^{(k)}} \frac{z^k}{k!},$$

and $a^{(k)} = \Gamma(a+k)/\Gamma(a)$.

Ref: N Simm and F Wei, On moments of the derivative of CUE characteristic polynomials and the Riemann zeta function, ***arXiv:2409.03687***, 2024.

An application to the radial distribution of zeros of $\Lambda_N'(z)$

Corollary

Let $n_N(r)$ be the number of zeros of $\Lambda_N'(z)$ inside the disc of radius r centered at the origin. Then uniformly with respect to r on any closed subset of $[0,1)$, we have

$$\lim_{N\to\infty} \int_0^r \frac{\mathbb{E}(n_N(t))}{t} dt = -\log(1-r^2)$$

and

$$\lim_{N\to\infty} \mathbb{E}(n_N(r)) = \frac{2r^2}{1-r^2}.$$

From the above Corollary, we also give a limit function for the radial density of the zeros of $\Lambda_N'(z)$. Specifically, we have that

$$\lim_{N\to\infty} \frac{d}{dr}\mathbb{E}(n_N(r)) = \frac{d}{dr}\left(\frac{2r^2}{1-r^2}\right) = \frac{4r}{(1-r^2)^2}.$$

This provides an alternative method to re-obtain Mezzadri's result [J. Phys. A, 36(12):2945-2962, 2003].

A connection to the moments of the derivatives of the Riemann zeta function off the critical line

Recall

	Random Matrix Theory	Number Theory
$\|z\| = 1$	$\mathbb{E}[\|\Lambda_N(z)\|^{2s}]$ $\sim g_s N^{s^2}$	$\frac{1}{T}\int_0^T \|\zeta(\frac{1}{2}+it)\|^{2s} dt$ $\sim a_s g_s (\log T)^{s^2}$
$\|z\| = 1$	$\mathbb{E}[\|\Lambda_N'(z)\|^{2s}]$ $\sim b_s N^{s^2+2s}$	$\frac{1}{T}\int_0^T \|\zeta'(\frac{1}{2}+it)\|^{2s} dt$ $\sim a_s b_s (\log T)^{s^2+2s}$

Here

$$a_s = \prod_p \left(1 - \frac{1}{p}\right)^{s^2} \sum_{m=0}^{\infty} \left(\frac{\Gamma(m+s)}{m! \cdot \Gamma(s)}\right)^2 p^{-m}.$$

	Random Matrix Theory	Number Theory
$\lvert z\rvert < 1$ $\sigma > \frac{1}{2}$	$\lim_{N\to\infty} \mathbb{E}[\lvert\Lambda_N(z)\rvert^{2s}]$ $= \dfrac{1}{(1-\lvert z\rvert^2)^{s^2}}$	$\lim_{T\to\infty} \frac{1}{T}\int_0^T \lvert\zeta(\sigma+\mathrm{i}t)\rvert^{2s} dt$ $\sim ?$
$\lvert z\rvert < 1$ $\sigma > \frac{1}{2}$	$\lim_{N\to\infty} \mathbb{E}[\lvert\Lambda_N'(z)\rvert^{2s}]$ $= \dfrac{d_s}{(1-\lvert z\rvert^2)^{s^2+2s}}$	$\lim_{T\to\infty} \frac{1}{T}\int_0^T \lvert\zeta'(\sigma+\mathrm{i}t)\rvert^{2s} dt$ $\sim ?$

Hardy and Littlewood(1923): for $\sigma > \frac{1}{2}$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T |\zeta(\sigma + \mathrm{i}t)|^{2s} dt$$

compute $s = 1, 2$.

Titchmarsh: Assume the truth of the Lindelöf hypothesis (i.e., $|\zeta(\frac{1}{2} + \mathrm{i}t)| = O(|t|^{\varepsilon})$ for any $\varepsilon > 0$), for any $s \in \mathbb{R}$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T |\zeta(\sigma + \mathrm{i}t)|^{2s} dt \sim \frac{a_s}{(2\sigma - 1)^{s^2}}, \quad \sigma \to \frac{1}{2}.$$

	Random Matrix Theory	Number Theory
$\lvert z\rvert < 1$ $\sigma > \frac{1}{2}$	$\lim_{N\to\infty}\mathbb{E}[\lvert\Lambda_N(z)\rvert^{2s}]$ $= \dfrac{1}{(1-\lvert z\rvert^2)^{s^2}}$	$\lim_{T\to\infty}\frac{1}{T}\int_0^T\lvert\zeta(\sigma+it)\rvert^{2s}dt$ $\sim \dfrac{a_s}{(2\sigma-1)^{s^2}}, \quad \sigma\to\dfrac{1}{2}$
$\lvert z\rvert < 1$ $\sigma > \frac{1}{2}$	$\lim_{N\to\infty}\mathbb{E}[\lvert\Lambda_N'(z)\rvert^{2s}]$ $= \dfrac{d_s}{(1-\lvert z\rvert^2)^{s^2+2s}}$	Conjecture: $\lim_{T\to\infty}\frac{1}{T}\int_0^T\lvert\zeta'(\sigma+it)\rvert^{2s}dt$ $\sim \dfrac{a_s d_s}{(2\sigma-1)^{s^2+2s}}, \quad \sigma\to\dfrac{1}{2}$

Theorem: Assume the truth of the Lindelöf hypothesis, the Conjecture holds for positive integer s.

We also compute $s = 1, 2$ unconditionally.

Microscopic regime

Theorem 3 (for finite matrix size N)

We have the following exact formula, valid for any $z \in \mathbb{C}$ and any positive integers N, s,

$$\mathbb{E}[|\Lambda_N'(z)|^{2s}] = \sum_{\lambda,\mu \in Y_s} \frac{f_\lambda f_\mu}{\lambda!\mu!} \det\left\{ \left(u^{\lambda_i+s-i} K_N^{(\lambda_i+s-i)}(u)\right)^{(\mu_j+s-j)} \right\}_{i,j=1}^{s}$$

where $u = |z|^2$ and

$$K_N(u) = \sum_{j=0}^{N+s-1} u^j.$$

Here

Y_n: the set of partitions λ satisfying $|\lambda| = n$.

f_λ: the number of standard Young tableaux of type λ,

$$f_\lambda = \frac{|\lambda|!}{\prod_{(i,j)\in\lambda} h(i,j)}.$$

$h(i,j)$: the hook length of (i,j).

Theorem 4 (Microscopic limit)

Let $|z|^2 = 1 - \frac{c}{N}$ for $c \in \mathbb{R}$ fixed. In particular, the case $|z| = 1$ corresponding to $c = 0$ is allowed. Then for any positive integer s, as $N \to \infty$ we have

$$\mathbb{E}[|\Lambda_N'(z)|^{2s}] \sim N^{s^2+2s} \sum_{\lambda,\mu \in Y_s} \frac{f_\lambda f_\mu}{\lambda!\mu!} \det\left(\int_0^1 x^{\lambda_i+\mu_j+2s-i-j} e^{-cx} dx\right)_{i,j=1}^{s}$$

We also obtain the equivalent expression, as $N \to \infty$,

$$\mathbb{E}(|\Lambda_N'(z)|^{2s}) \sim N^{s^2+2s} \frac{\partial^{2s}}{\partial v^s \partial w^s} \det\left\{\frac{\partial^{i+j-2} F_c(v,w)}{\partial v^{i-1} \partial w^{j-1}}\right\}_{i,j=1}^{s} \Bigg|_{v=w=0},$$

where

$$F_c(v,w) = \int_0^1 J_0(2\sqrt{vx}) J_0(2\sqrt{wx}) e^{-cx} dx,$$

and J_0 is the Bessel function of the first kind. Furthermore, the leading terms on the right-hand sides of the above asymptotics are strictly positive.

Commented by Bothner, the expression of F_c can be recognised as a particular case of the finite temperature Bessel kernel. When $c = 0$, this is exactly the Bessel kernel.

Commented by Akemann, the expression of F_c has appeared in the RMT application to QCD (that is, Quantum Chromodynamics) with imaginary chemical potential.

3. Proof Sketch of the Main Results

Approaches to Theorem 1

Step 1: Prove the following convergence for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > -1$,

$$\lim_{N\to\infty} \mathbb{E}(|\Lambda_N'(z)|^{2s}) = \mathbb{E}(|\Lambda'(z)|^{2s}),$$

uniformly holds for any closed subset of $\{z : |z| < 1\}$, where

$$\Lambda(z) = e^{G(z)} = e^{\sum_{k=1}^{\infty} \frac{\mathcal{N}_k}{\sqrt{k}} z^k}$$

and $\{\mathcal{N}_k\}_{k=1}^{M}$ are i.i.d. standard complex normal random variables with

$$\{\mathrm{Tr}(U^{-k})\}_{k=1}^{M} \xrightarrow{d} \{\mathcal{N}_k\}_{k=1}^{M}, \qquad N \to \infty,$$

implied by the strong Szegö limit theorem for Toeplitz determinants.

Step 2: Compute $\mathbb{E}(|\Lambda'(z)|^{2s})$.

Sketch of the proof of Step 1 for Theorem 1

It follows from the the following result on the uniform integrability of $\{|\Lambda_N(z)|^{2s}\}_N$.

Lemma (Uniform integrability)

Assume that $s \in \mathbb{C}$ *with* $\mathrm{Re}(s) > -1$*. Consider the quantity*

$$X_N = |\Lambda_N'(z)|^{2s}.$$

Then for any $\delta > 0$ *fixed,* $|z| < 1 - \delta$ *and for* $\varepsilon > 0$ *small enough and for* N *sufficiently large, there exists a constant* $C > 0$ *independent of* N *and* z *such that*

$$\mathbb{E}(|X_N|^{1+\varepsilon}) < C.$$

Proposition (Bounds on negative moments)

For any a with $0 \le a < 2$, *let* $r = |z| < 1$ *and* $N > 4$. *Then there is a constant C depending only on a and r such that*

$$\mathbb{E}(|\Lambda_N'(r)|^{-a}) \le C.$$

By Hölder's inequality,

$$\mathbb{E}(|\Lambda_N'(r)|^{-a}) \le \mathbb{E}\left(\left|\frac{\Lambda_N'(r)}{\Lambda_N(r)}\right|^{-aq}\right)^{1/q} \mathbb{E}\left(|\Lambda_N(r)|^{-a\ell}\right)^{1/\ell}$$

where $q = \frac{\ell}{\ell-1} > 1$ and we choose ℓ large enough such that $aq < 2$.

Forrester and Keating: the boundness of $\mathbb{E}\left(|\Lambda_N(r)|^{-a\ell}\right)$.

It reduced to proving the following boundedness for $0 \le a < 2$,

$$\mathbb{E}\left(\left|\frac{\Lambda_N'(r)}{\Lambda_N(r)}\right|^{-a}\right) \le C.$$

A linear statistics

$$\frac{\Lambda_N'(r)}{\Lambda_N(r)} = -\sum_{j=1}^{N} f(\theta_j),$$

where

$$f(\theta) = \frac{e^{-\mathrm{i}\theta}}{1 - re^{-\mathrm{i}\theta}} = \sum_{n=1}^{\infty} r^{n-1} e^{-\mathrm{i}n\theta}.$$

By integrating by parts, it reduced to proving for $0 \leq a < 2$,

$$\int_0^1 y^{-a-1} \mathbb{P}\left(\left| \frac{\Lambda_N'(r)}{\Lambda_N(r)} \right| < y \right) dy \leq C$$

So it suffices to show

$$\mathbb{P}\left(\left| \frac{\Lambda_N'(r)}{\Lambda_N(r)} \right| < y \right) \leq Cy^2.$$

Proposition (Small deviations inequality (Halász,1977))

$$\mathbb{P}\left(\left|\frac{\Lambda_N'(r)}{\Lambda_N(r)}\right| < y\right)$$

$$\leq \quad y^2 \int_{|\xi_1|<y^{-1}} \int_{|\xi_2|<y^{-1}} d\xi_1\, d\xi_2 \left|\mathbb{E}\left(e^{\mathrm{i}\xi_1 \mathrm{Re}\left(\frac{\Lambda_N'(r)}{\Lambda_N(r)}\right) + \mathrm{i}\xi_2 \mathrm{Im}\left(\frac{\Lambda_N'(r)}{\Lambda_N(r)}\right)}\right)\right|,$$

Split the above integral over three regions:

$$\begin{aligned} R_1 &= \{(\xi_1,\xi_2) : |\xi_1| + |\xi_2| < N\} \\ R_2 &= \{(\xi_1,\xi_2) : N \leq |\xi_1| + |\xi_2| < N^8\} \\ R_3 &= \{(\xi_1,\xi_2) : |\xi_1| + |\xi_2| > N^8\} \end{aligned}$$

Estimates on the regions R_1, R_2 and R_3.

For $(\xi_1, \xi_2) \in R_1 \cup R_2$,

Johansson(1997): Change of variables method + the inequality version of the strong Szegö limit theorem as follows,

$$\mathbb{E}[e^{\sum_{j=1}^N g(\theta_j)}] \leq e^{N\hat{g}_0 + \sum_{k\geq 1} k|\hat{g}_k|^2}$$

for a real-valued function $g(\theta)$.

Lemma

Let $(\xi_1, \xi_2) \in R_1$. There is a constant $c > 0$ depending only on r such that

$$\left|\mathbb{E}\left(e^{\mathrm{i}\xi_1 \mathrm{Re}\left(\frac{\Lambda_N'(r)}{\Lambda_N(r)}\right) + \mathrm{i}\xi_2 \mathrm{Im}\left(\frac{\Lambda_N'(r)}{\Lambda_N(r)}\right)}\right)\right| \leq e^{-c\xi_1^2 - c\xi_2^2}.$$

Let $(\xi_1, \xi_2) \in R_2$, we have

$$\left|\mathbb{E}\left(e^{\mathrm{i}\xi_1 \mathrm{Re}\left(\frac{\Lambda_N'(r)}{\Lambda_N(r)}\right) + \mathrm{i}\xi_2 \mathrm{Im}\left(\frac{\Lambda_N'(r)}{\Lambda_N(r)}\right)}\right)\right| \leq e^{-cN^2}.$$

For $(\xi_1, \xi_2) \in R_3$,

A Toeplitz determinant representation

$$\left| \mathbb{E}\left(e^{\mathrm{i}\xi_1 \mathrm{Re}\left(\frac{\Lambda_N'(r)}{\Lambda_N(r)}\right) + \mathrm{i}\xi_2 \mathrm{Im}\left(\frac{\Lambda_N'(r)}{\Lambda_N(r)}\right)} \right) \right| = \det\left\{ \hat{h}_{j-k} \right\}_{j,k=0}^{N-1}$$

By Hadamard's inequality, the above

$$\leq \prod_{j=1}^{N} \left(\sum_{k=1}^{N} |\hat{h}_{j-k}|^2 \right)^{\frac{1}{2}}.$$

Then apply the stationary phase approximation to $\hat{h}_k$.

Lemma

Let $(\xi_1, \xi_2) \in R_3$. Then there is a constant $C > 0$ depending only on r such that for all N we have

$$\left| \mathbb{E}\left(e^{\mathrm{i}\xi_1 \mathrm{Re}\left(\frac{\Lambda_N'(r)}{\Lambda_N(r)}\right) + \mathrm{i}\xi_2 \mathrm{Im}\left(\frac{\Lambda_N'(r)}{\Lambda_N(r)}\right)} \right) \right| \leq C^N N^{-N/2} (|\xi_1| + |\xi_2|)^{-N/4}.$$

Proof sketch of Step 2 in the approach to Theorem 1

We now compute $\mathbb{E}(|\Lambda'(z)|^{2s})$ for $\mathrm{Re}(s) > -1$.

Recall that

$$\Lambda'(z) = G'(z)e^{G(z)},$$

where

$$G(z) = \sum_{k=1}^{\infty} \frac{\mathcal{N}_k}{\sqrt{k}} z^k.$$

and

$$G'(z) = \sum_{k=1}^{\infty} \sqrt{k}\mathcal{N}_k z^{k-1}$$

About the multivariate complex Gaussian vector $(G(z), G'(Z))$:

- The mean vector and the relation matrix are 0.

- The covariance matrix is

$$\begin{aligned}\Gamma &= \begin{pmatrix} \mathbb{E}(|G(z)|^2) & \mathbb{E}(G(z)\overline{G'(z)}) \\ \mathbb{E}(\overline{G(z)}G'(z)) & \mathbb{E}(|G'(z)|^2) \end{pmatrix} \\ &= \begin{pmatrix} -\log(1-|z|^2) & \frac{z}{1-|z|^2} \\ \frac{\bar{z}}{1-|z|^2} & \frac{1}{(1-|z|^2)^2} \end{pmatrix}\end{aligned}$$

- The joint density function

$$f(w_1, w_2) = \frac{1}{\pi^2 \det(\Gamma)} \exp\left(-\mathbf{w}^T \Gamma^{-1} \mathbf{w}\right)$$

with $\mathbf{w} = (w_1, w_2)^T$.

$$\mathbb{E}(|\Lambda'(z)|^{2s}) = \int_{\mathbb{C}} d^2 w_2 \int_{\mathbb{C}} d^2 w_1 \, |w_2|^{2s} \, e^{s w_1 + s\overline{w_1}} \, f(w_1, w_2).$$

We first do the integral with respect to w_1 and then do the integral with respect to w_2. It is then reduced to computing the following integral

$$\frac{1}{\pi} \int_{\mathbb{C}} d^2 w_2 |w_2|^{2s} e^{-|w_2|^2 + s z w_2 + s\overline{z w_2}}$$

with $z = \frac{\Gamma_{1,2}}{\sqrt{\Gamma_{2,2}}}$. Expand inside the exponential, it is

$$\begin{aligned}
&\sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{(sz)^{k_1} (s\overline{z})^{k_2}}{(k_1)!(k_2)!} \frac{1}{\pi} \int_{\mathbb{C}} d^2 w_2 |w_2|^{2s} e^{-|w_2|^2} (w_2)^{k_1} (\overline{w_2})^{k_2} \\
&= \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{(sz)^{k_1} (s\overline{z})^{k_2}}{(k_1)!(k_2)!} \delta_{k_1,k_2} \Gamma\left(s + \frac{k_1+k_2}{2} + 1\right) \\
&= \sum_{k=0}^{\infty} \frac{\Gamma(s+k+1)}{(k!)^2} s^{2k} |z|^{2k} \\
&= \Gamma(s+1) \, {}_1F_1(s+1, 1; s^2 |z|^2).
\end{aligned}$$

Sketch the proof in the approach to Theorems 3 and 4

Theorem (Akemann and Vernizzi'00)

For $s \in \mathbb{N}$, the average of a product of $2s$ characteristic polynomials is

$$\mathbb{E}\left[\prod_{j=1}^{s} \det(I - z_j U)\det(I - w_j U^*)\right] = \frac{\det\left\{\sum_{l=0}^{N+s-1}(z_i w_j)^l\right\}_{i,j=1}^{s}}{\prod_{1\le i<j\le s}(z_j - z_i)\prod_{1\le i<j\le s}(w_j - w_i)}.$$

$$\mathbb{E}\left[|\Lambda_N'(z)|^{2s}\right] = \prod_{j=1}^{s}\frac{\partial}{\partial z_j}\frac{\partial}{\partial w_j}\mathbb{E}\left[\prod_{j=1}^{s}\det(I - z_j U)\det(I - w_j U^*)\right]\Bigg|_{\mathbf{z}=\mathbf{w}=|z|}.$$

Our result

Proposition (A proposition on the merge process)

Let $n \geq 1$ be an integer. Let $f(x_1, \ldots, x_n)$ be a multivariate anti-symmetric polynomial, that is, for any permutation σ of $\{1, 2, \ldots, n\}$,

$$f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = \mathrm{sign}(\sigma) f(x_1, \ldots, x_n).$$

Then

$$\prod_{i=1}^{n} \frac{\partial}{\partial x_i} \frac{f(x_1, \ldots, x_n)}{\prod_{1 \leq i < j \leq n}(x_i - x_j)} \Big|_{x_1 = \cdots = x_n = x}$$

$$= \sum_{\lambda \in Y_n} \frac{f_\lambda}{\prod_{i=1}^{n}(\lambda_i + n - i)!} \prod_{i=1}^{n} \frac{\partial^{\lambda_i + n - i}}{\partial x_i^{\lambda_i + n - i}} f(x_1, \ldots, x_n) \Big|_{x_1 = \cdots = x_n = x}.$$

Recall: Y_n: the set of partitions λ satisfying $|\lambda| = n$.
f_λ: the number of standard Young tableaux of type λ,

$$f_\lambda = \frac{|\lambda|!}{\prod_{(i,j) \in \lambda} h(i,j)}.$$

$h(i,j)$: the hook length of (i,j).

Note that

$$K_N(u) = \frac{1-u^{N+s}}{1-u} = (N+s)\int_0^1 (1-x(1-u))^{N+s-1}dx.$$

Let $u = 1-\frac{c}{N}$. The leading coefficient of $\mathbb{E}[|\Lambda_N'(1-\frac{c}{N})|^{2s}]$ is

$$b_s(c) := \sum_{\lambda,\mu\in Y_s} \frac{f_\lambda f_\mu}{\lambda!\mu!} \det\left\{\int_0^1 x^{2s+\lambda_i-i+\mu_j-j} e^{-cx} dx\right\}_{i,j=1}^{s}.$$

By the Andréief identity and using the Schur polynomials,

$$b_s(c) = \frac{1}{s!}\int_{[0,1]^s}\left(\sum_{\lambda\in Y_s}\frac{f_\lambda}{\lambda!}s_\lambda(\mathbf{x})\right)^2 \prod_{j=1}^{s} e^{-cx_j}\Delta(\mathbf{x})^2 d\mathbf{x}.$$

By Schur positivity, $b_s(c) > 0$.

The known relation between the hook length and the Schur polynomial evaluated at $1_s = (1, \ldots, 1)$ with 1 appearing s times, in the form

$$f_\lambda = \frac{s_\lambda(1_s)}{\lambda!} \prod_{j=0}^{s} j!.$$

Then the sum in $b_s(c)$ is

$$\sum_{\lambda \in Y_s} \frac{f_\lambda}{\lambda!} s_\lambda(\mathbf{x}) = \left(\prod_{j=0}^{s} j! \right) \frac{(-1)^s}{s!} \frac{\partial^s}{\partial v^s} \sum_{\lambda, l(\lambda) \leq s} \frac{s_\lambda(1_s)}{(\lambda!)^2} s_\lambda(\mathbf{x})(-v)^{|\lambda|} \bigg|_{v=0},$$

We now replace $s_\lambda(1_s)(-v)^{|\lambda|}$ with $s_\lambda(-\mathbf{v})$ where $\mathbf{v}$ consists of s new variables. Due to homogeneity of the Schur polynomials, we then recover the desired quantity after taking $\mathbf{v} = (v, v, \ldots, v)$ in the end.

Using the Cauchy-Binet identity, we have

$$\sum_{\lambda, l(\lambda)\leq s} \frac{s_\lambda(\mathbf{x}) s_\lambda(\mathbf{v})}{(\lambda!)^2} = \frac{\det\left\{\sum_{\ell=0}^{\infty} \frac{(-x_i v_j)^\ell}{(\ell!)^2}\right\}_{i,j=1}^{s}}{\Delta(\mathbf{x})\Delta(\mathbf{v})},$$

The function inside the above determinant is the series definition of the Bessel function of the first kind

$$J_0(2\sqrt{x}) = \sum_{j=0}^{\infty} \frac{(-x)^j}{(j!)^2}.$$

Recently, Akemann, Kieburg et al. a Borel transformation of the inital CUE kernel to obtain the Bessel function alternatively.

Further questions

A question related to Painlevé equations

Without taking the derivative,

$$\mathbb{E}[|\Lambda_N(z)|^{2s}]$$

for $|z| = 1$, keating and Snaith(2000), Selberg integral;

for $|z| < 1$, for fixed matrix size N,

Deaño and Simm (2022):

$$\mathbb{E}[|\Lambda_N(z)|^{2s}] \qquad \mathrm{Re}(s) > -1$$

$$= \frac{1}{(2\pi \mathrm{i})^N N!} \int_{\{z:|z|=1\}^N} \prod_{j=1}^{N} \frac{dw_j}{w_j} w_j^{-s/2} |1 + w_j|^s (1 + z^2 w_j)^s \prod_{1 \le i < j \le N} |w_i - w_j|^2$$

Forrester and Witte(2004):

$$\mathbb{E}[|\Lambda_N(z)|^{2s}] = (1 - |z|^2)^{-s^2} \exp\left(- \int_{1-|z|^2}^{1} \frac{\sigma_{N,s}^{(\mathrm{VI})}(t) - c_1^2 t + \frac{c_1^2 + c_2^2}{2}}{t(1-t)} dt \right),$$

where $\sigma_{N,s}^{(\mathrm{VI})}(t)$ satisfies the Jimbo-Miwa-Okamoto σ-form of the Painlevé VI equation, $c_1 = s + N/2$ and $c_2 = N/2$.

Actually, for integer s,

$$\mathbb{E}[|\Lambda_N(z)|^{2s}] = (1-|z|^2)^{-s^2}\mathbb{P}(t_{\max}^{(0,N)} \le 1-|z|^2),$$

the largest eigenvalue distribution in the Jacobi ensemble.

Microscopic limit for $|z|^2 = 1 - \frac{c}{N}$ with $c > 0$,

Rewrite $\mathbb{E}[|\Lambda_N(z)|^{2s}]$ as a Toeplitz determinant,

$$z^{sN} \det\left\{ \frac{1}{2\pi} \int_0^{2\pi} m(e^{\mathrm{i}\theta}, z) e^{\mathrm{i}(k-j)\theta} d\theta \right\}_{k,j=0}^{N-1}$$

with symbol

$$m(e^{\mathrm{i}\theta}, z) = (e^{\mathrm{i}\theta} - z)^s \left(e^{\mathrm{i}\theta} - \frac{1}{z}\right)^s e^{-\mathrm{i}\theta s} e^{-\mathrm{i}\pi s}.$$

Claeys, Its and Krasovsky(2011), Forrester and Witte(2002),

$$\mathbb{E}[|\Lambda_N(z)|^{2s}] \sim (N/c)^{s^2} \exp\left(-\int_c^\infty \frac{\sigma_s^{(\mathrm{V})}(t)}{t} dt\right), \quad \mathrm{Re}(s) > -\frac{1}{2}$$

with $|z|^2 = 1 - \frac{c}{N}$, where $\sigma_s^{(\mathrm{V})}(t)$ satisfies the Jimbo-Miwa-Okamoto σ-form of the Painlevé V equation

$$(t\sigma'')^2 - [\sigma - t\sigma' + 2(\sigma')^2 + 2s\sigma']^2 + (4\sigma')^2(s+\sigma')^2 = 0.$$

Summarize the above,

$\lvert z\rvert < 1$	for finite matrix size N	Microscopic limit
$\mathbb{E}[\lvert\Lambda_N(z)\rvert^{2s}]$	σ-Painlevé VI	σ-Painlevé V
$\mathbb{E}[\lvert\Lambda'_N(z)\rvert^{2s}]$	?	?

It is known by Forrester and Witte (2006), Basor, Bleher, Buckingham, Grava, Its, Its, and Keating (2019), Keating and Wei (2023), Assoitis, Gunes, Keating and Wei (2024),

$\lvert z\rvert = 1$	for finite matrix size N	large N-limit
$\mathbb{E}[\lvert\Lambda'_N(z)\rvert^{2s}]$	σ-Painlevé V	σ-Painlevé III'
$\mathbb{E}[\lvert\Lambda_N^{(n)}(z)\rvert^{2s}]$ $n \geq 2$	the derivatives of σ-Painlevé V	the derivatives of σ-Painlevé III'

Thank you and

Happy Birthday, Peter!