The multiplicative constant in large gap asymptotics of unitary random matrix ensembles near critical points

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Plan

- Gap probability and large gap asymptotics near regular points
- Gap probability and large gap asymptotics near critical points
- Relation to integrable PDEs

Random unitary ensembles

- Consider the random unitary ensemble
$$\frac{1}{Z_n} e^{-n\mathrm{Tr}V(M)} dM, \tag{1}$$
defined on $n \times n$ Hermitian matrices M.
- The potential V is a real analytic function over $\mathbb{R}$ satisfying
$$\lim_{|x|\to\infty} \frac{V(x)}{\log(1+x^2)} = +\infty.$$
- It is well-known that the eigenvalues of M form a determinantal point process characterized by the correlation kernel
$$K_n(x,y) := e^{-\frac{n}{2}(V(x)+V(y))} \sum_{i=0}^{n-1} p_i(x)p_i(y).$$
- When $V(x) = \frac{x^2}{2}$, this is the well-known Gaussian unitary ensemble (GUE).

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Eigenvalue distribution

- The limiting mean density of eigenvalues are

$$\lim_{n\to\infty} \frac{1}{n} K_n(x,x) = \rho_V(x).$$

- $d\mu_V = \rho_V(x)dx$ is the unique equilibrium measure that minimizes the logarithmic energy functional

$$I_V(\mu) = \iint \log \frac{1}{|x-y|} d\mu(x)d\mu(y) + \int V(x)d\mu(x),$$

among all the probability measure μ on $\mathbb{R}$.

- Euler-Lagrange variational conditions: there exists a constant ℓ_V such that

$$2\int \log|x-y|d\mu_V(y) - V(x) + \ell_V = 0, \quad x \in \mathrm{supp}(\mu_V), \tag{2}$$

$$2\int \log|x-y|d\mu_V(y) - V(x) + \ell_V \le 0, \quad x \in \mathbb{R}. \tag{3}$$

Classical universality classes

- Generically, $\rho_V(x)$ is positive in $\text{supp}(\mu_V)$, and vanishes as a squared root at the endpoints.
- Soft-edge universality: let b_V be the endpoint of $\text{supp}(\mu_V)$, there exists a constant c_V such that

$$\lim_{n\to\infty} \frac{1}{c_V n^{2/3}} K_n\left(b_V + \frac{u}{c_V n^{2/3}}, b_V + \frac{v}{c_V n^{2/3}}\right) = K^{\text{Ai}}(u,v).$$

- Bulk universality: for any x^* such that $\rho_V(x^*) > 0$, the limiting correlation kernel is the sine kernel

$$\lim_{n\to\infty} \frac{1}{n\rho_V(x^*)} K_n\left(x^* + \frac{u}{n\rho_V(x^*)}, x^* + \frac{v}{n\rho_V(x^*)}\right) = K^{\sin}(u,v).$$

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Non-standard universality classes

For general real analytic potentials $V(x)$, there are three types of critical points:

Deift, Kriecherbauer and McLaughlin (JAT, 1998)

Kuijlaars and McLaughlin (CPAM, 2000)

- Critical edge point: an endpoint x^* of $\mathrm{supp}(\mu_V)$ where $\rho_V(x)$ vanishes to an order higher than square root:

$$\rho_V(x) \sim |x - x^*|^{2k+\frac{1}{2}}, \qquad \text{as } x \to x^*,\ k \in \mathbb{N}^+.$$

- Critical interior point: a point $x^* \in \mathrm{supp}(\mu_V)$ where $\rho_V(x)$ vanishes with even order:

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Kernels for non-standard universality classes

- Critical edge point: the limit of the correlation kernel is built from functions relevant to the Painlevé I hierarchy: Brézin, Marinar and Parisi (PLB, 1990)

  Bowick and Brézin (PLB, 1991), Claeys and Vanlessen (CMP, 2007)

$$\lim_{n\to\infty} \frac{1}{c_V n^{2/(4k+3)}} K_n\left(b_V + \frac{u}{c_V n^{2/(4k+3)}}, b_V + \frac{v}{c_V n^{2/(4k+3)}}\right) = K_{\mathrm{P_I}}^{(k)}(u, v; x).$$

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  For $k = 1$, this result was rigorously established. Bleher and Its (CPAM, 2003)

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- Critical exterior point: finite size GUE kernels.

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Gap probability: soft edge

- Let λ_n be the largest eigenvalue of M, then

$$\lim_{n\to\infty} \text{Prob}\left(c_V n^{\frac{2}{3}}(\lambda_n - b_V) < s\right) = \det(I - \mathbb{K}_s^{\text{Ai}}),$$

where $\mathbb{K}_s^{\text{Ai}}$ is the trace-class operator acting on $L^2(s,\infty)$ with the Airy kernel.

- Tracy-Widom distribution: Tracy and Widom (CMP, 1994)

$$F_{\text{TW}}(s) := \det(I - \mathbb{K}_s^{\text{Ai}}) = \exp\left(-\int_s^\infty (\tau - s) y_{HM}^2(\tau) d\tau\right).$$

- $y_{HM}(x)$ is the Hastings-McLeod solution to the Painlevé II (PII) equation

$$y''(x) = xy(x) + 2y^3(x),$$

with $y_{HM}(x) \sim \sqrt{-x/2}$, $x \to -\infty$ and $y_{HM}(x) \sim \text{Ai}(x)$, $x \to +\infty$.

- Large gap asymptotics: Deift, Its and Krasovsky (CMP, 2008)

$$\log F_{\text{TW}}(s) = \frac{s^3}{12} - \frac{1}{8}\log(-s) + \frac{1}{24}\log 2 + \zeta'(-1) + O(|s|^{-3/2}),$$

as $s \to -\infty$, where $\zeta(s)$ is the Riemann zeta-function.

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Gap probability: bulk

- $\lim_{n\to\infty} \text{Prob}\Big(M \text{ has no eigenvalues in } (x^* - \frac{s}{n\rho_V(x^*)}, x^* + \frac{s}{n\rho_V(x^*)})\Big) = \det(I - \mathbb{K}^{\sin}_{(-s,s)})$, where $\mathbb{K}^{\sin}_{(-s,s)}$ is the trace-class operator acting on $L^2(-s,s)$ with the sine kernel.
- The sine-kernel determinant admits an explicit integral expression in terms of the σ-form of the Painlevé V equation Jimbo, Miwa, Mori and Sato (Phys. D 1980)

$$\det(I - \mathbb{K}^{\sin}_{(-s,s)}) = \exp\left(\int_0^s \frac{\sigma_V(\tau)}{\tau} d\tau\right),$$

where $\sigma_V(x)$ is a special solution of the following equation

$$(x\sigma_V'')^2 + 4\Big(4\sigma_V - 4x\sigma_V' - {\sigma_V'}^2\Big)(\sigma_V - x\sigma_V') = 0, \tag{4}$$

with $\sigma_V(x) \sim -\frac{2x}{\pi}, x \to 0^+$ and $\sigma_V(x) \sim -x^2 - \frac{1}{4}, x \to +\infty$.

- As $s \to \infty$, Krasovsky (IMRN 2004), Ehrhardt (CMP, 2006)

$$\log\det(I - \mathbb{K}^{\sin}_{(-s,s)}) = -\frac{s^2}{2} - \frac{1}{4}\log s + \frac{1}{12}\log 2 + 3\zeta'(-1) + O(s^{-1}).$$

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Gap probability: more

- Forrester-Chen-Eriksen-Tracy conjecture: Forrester (Nucl. Phys. B, 1993) Chen, Eriksen and Tracy (J. Phys. A, 1995)

  If the density of state behaves as $|x - x^*|^\kappa$ near a point x^*, then the probability $E(s)$ of emptiness of the (properly scaled) interval $(x^* - s, x^* + s)$ behaves like

$$E(s) \sim \exp(-Cs^{2\kappa+2}), \qquad s \to +\infty.$$

- The constant terms in the large gap asymptotics are difficult to derive.
- The study of gap probability has a long history and many results have been obtained in the literature.

  P. J. Forrester, Asymptotics of spacing distributions 50 years later, Math. Sci. Res. Inst. Publ., 65 Cambridge University Press, New York, 2014, 199–222.
- During the past ten years, there are still exciting developments, for example,
  - gap probability on two large intervals; Fahs, Krasovsky, Maroudas
  - gap probability for thinned determinantal point processes; Charlier, Claeys, D., Liu, Yao, Xu, Zhang
  - gap probability in 2D models; Byun, Charlier, Park
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Critical edge point: higher-order analogues of the TW distribution

- Let $\mathbb{K}_s^{\mathrm{P}_\mathrm{I}^\mathrm{k}}$ be the trace-class operator acting on $L^2(s,\infty)$ with the $\mathrm{P}_\mathrm{I}^\mathrm{k}$ kernel, then $\det(I-\mathbb{K}_s^{\mathrm{P}_\mathrm{I}^\mathrm{k}})$ can be considered as the higher-order analogues of the Tracy-Widom distribution. It can be explicitly expressed in terms of a special smooth solution to the Painlevé II hierarchy. Claeys, Its and Krasovsky (CPAM, 2010)
- As $s \to -\infty$, we have

$$\log\det(I-\mathbb{K}_s^{\mathrm{P}_\mathrm{I}^\mathrm{k}}) = \frac{1}{4(4k+3)}\alpha_k^2 s^{4k+3} + \frac{\alpha_k}{2(2k+2)} x s^{2k+2} + \sum_{m=2}^{4k+1} a_m |s|^m + \frac{x^2 s}{4} - \frac{2k+1}{8}\log|s| + C^{(k)} + O(|s|^{-2}),$$

where a_m are functions of $x, t_1, \ldots, t_{2k-1}$ and vanish when $t_1 = \ldots = t_{2k-1} = 0$,

$$\alpha_k := \frac{2\Gamma\left(2k+\frac{3}{2}\right)}{\Gamma(2k+2)\,\Gamma\left(\frac{3}{2}\right)}, \qquad (5)$$

- The s-independent constant $C^{(k)}$ is unknown except for $k=0$:

$$\det(I-\mathbb{K}_s^{\mathrm{P}_\mathrm{I}^0}) = F_{\mathrm{TW}}(2^{\frac{2}{3}} s).$$

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Our result: large gap asymptotics

Theorem (D., Long, Xu, Yao and Zhang, arXiv:2501.12679)

For $k = 1, 2, \ldots$, let $\mathbb{K}_s^{\mathrm{P}_{\mathrm{I}}^{\mathrm{k}}}$ be the trace-class operator acting on $L^2(s, \infty)$ with the kernel $K^{(k)}(u, v; x)$ and define

$$F_{\mathrm{I}}(s; x) := \log\det(I - \mathbb{K}_s^{\mathrm{P}_{\mathrm{I}}^{\mathrm{k}}}).$$

Then, we have, as $s \to -\infty$,

$$\begin{aligned} F_{\mathrm{I}}(s;x) = \frac{1}{4(4k+3)}\alpha_k^2 s^{4k+3} + \frac{\alpha_k}{2(2k+2)} x s^{2k+2} + \frac{x^2 s}{4} - \frac{1}{8}\log\left|\alpha_k s^{2k+1} + x\right| \\ -I_h(x) + \frac{(2k+1)^2}{2(2k+2)(4k+3)}\alpha_k^{-\frac{1}{2k+1}} \cdot x^{\frac{4k+3}{2k+1}} + \frac{k\log(x^2+1)}{24(2k+1)} \\ + \frac{\log(2k+1)}{24} + \frac{\log\alpha_k}{24(2k+1)} + \frac{1}{24}\log 2 + \zeta'(-1) + \mathcal{O}(|s|^{-\epsilon_0}). \end{aligned}$$

Our result: large gap asymptotics

Theorem (Continued)

The above result is uniformly valid for $x \in \left[-c_1|s|^{2k+1}, \alpha_k|s|^{2k+1} - c_2|s|^{\frac{2k}{3}+\epsilon}\right]$ *with* c_1, c_2 *being arbitrarily fixed real positive numbers and fixed* $\epsilon \in (0, \frac{4k}{3}+1)$*, where* $\epsilon_0 = \min\{\frac{1}{6}, 2\epsilon\}$*. Here, the constant* α_k *is defined in* (5) *and*

$$I_h(x) := -\int_{-\infty}^{x} \left[h_{\mathrm{I}}(\tau) - h_{\mathrm{I,asy}}(\tau)\right] d\tau = \int_{x}^{+\infty} \left[h_{\mathrm{I}}(\tau) - h_{\mathrm{I,asy}}(\tau)\right] d\tau,$$

where $h_I(x)$ *is the Hamiltonian associated with the special solution* $\mathsf{q}(x)$ *of the Painlevé I hierarchy* $\mathrm{P}_{\mathrm{I}}^{2\mathrm{k}}$ *in* (6) *and*

$$h_{\mathrm{I,asy}}(x) := \frac{(2k+1)}{2(2k+2)}\alpha_k^{-\frac{1}{2k+1}} \cdot x^{\frac{2k+2}{2k+1}} + \frac{kx}{12(2k+1)(x^2+1)}.$$

The Painlevé I hierarchy

- The m-th member of $\mathrm{P}_{\mathrm{I}}^{\mathrm{m}}$ is a nonlinear ordinary differential equation of order $2m$ defined by

$$x + \mathcal{L}_m(q) + \sum_{j=1}^{m-1} t_j \mathcal{L}_{j-1}(q) = 0, \qquad t_1, \ldots, t_{m-1} \in \mathbb{R}, \tag{6}$$

where the operator $\mathcal{L}$ satisfies the Lenard recursion relation

$$\begin{cases} \frac{d}{dx}\mathcal{L}_{k+1}(q) = \left(\frac{1}{4}\frac{d^3}{dx^3} - 2q\frac{d}{dx} - q_x\right)\mathcal{L}_k(q), \quad k = 0, \ldots, m-1, \\ \mathcal{L}_0(q) = -4q, \quad \mathcal{L}_j(0) = 0, \quad j = 1, \ldots, m. \end{cases}$$

- If $m = 1$, one recovers the Painlevé I equation $q_{xx} = 6q^2 + x$.
- If $m = 2$, we have

$$q_{xxxx} = 4x - 40q^3 + 10q_x^2 + 20qq_x - 16t_1 q.$$

A special solution to the Painlevé I hierarchy

- There exists a real and pole-free solution $\mathsf{q}(x)$ to each equation $\mathrm{P}_{\mathrm{I}}^{2\mathrm{k}}$ with the boundary condition Claeys (Phys. D, 2012)

$$\mathsf{q}(x) = \frac{1}{2}\alpha_k^{-\frac{1}{2k+1}} x^{\frac{1}{2k+1}} + \mathcal{O}\left(|x|^{-\frac{1}{2k+1}}\right), \qquad x \to \pm\infty.$$

- The Hamiltonian $h_{\mathrm{I}}(x) = h_{\mathrm{I}}(x, t_1, \dots, t_{2k-1})$ is related to $\mathsf{q}(x)$ through the relation $dh_{\mathrm{I}}(x)/dx = \mathsf{q}(x)$.

- For $\mathrm{P}_{\mathrm{I}}^{2}$, $\mathsf{q}(x)$ is called the tritronquée solution.

  Grava, Kapaev and Klein (CA, 2015)

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Gap probability near the critical interior point

- Let $\mathbb{K}_s^{\mathrm{P}_{\mathrm{II}}^{\mathrm{k}}}$ be the trace-class operator acting on $L^2(-s,s)$ with the $\mathrm{P}_{\mathrm{II}}^{\mathrm{k}}$ kernel. In the first bulk critical case ($k=1$), as $s \to +\infty$ Bothner and Its (CMP, 2014)

$$\begin{aligned}\ln\det(I-\mathbb{K}_s^{\mathrm{P}_{\mathrm{II}}}) &= -\frac{2}{3}s^6 - s^4x - \frac{1}{2}(sx)^2 - \frac{3}{4}\log s \\ &\quad + \int_x^{+\infty}(\tau - x)q_{HM}^2(\tau)d\tau - \frac{1}{6}\log 2 + 3\zeta'(-1) + O(s^{-1}).\end{aligned}$$

- An algebraic singularity $|\det M|^{2\alpha}$ with $\alpha > -1/2$ can be introduced in the model. If the critical interior point is located at the origin, we have the general $\mathrm{P}_{\mathrm{II}^\alpha}$ kernel.
Claeys, Kuijlaars and Vanlessen (Ann. Math., 2008)

- The large gap asymptotics for the general $\mathrm{P}_{\mathrm{II}^\alpha}$ kernel were established in Xu and D. (CMP, 2019). In addition, we get an explicit integral representation:

$$\begin{aligned}&\det(I-\mathbb{K}_{(-s,s)}^{\mathrm{P}_{\mathrm{II}}}) \\ &= \exp\left(-\int_{-\infty}^{x}\left(q^2(\tau;\alpha) - 2^{-2/3}w_2^2(-2^{-1/3}\tau - 2^{2/3}s^2; -2^{2/3}s^2,\alpha)\right)(\tau - x)d\tau\right),\end{aligned}$$

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For $k = 1, 2, \ldots$, let $\mathbb{K}_s^{\mathrm{P}_{\mathrm{II}}^{\mathrm{k}}}$ be the trace-class operator acting on $L^2(-s, s)$ with the kernel $K^{(k)}(u, v; x)$ and define

$$F_{\mathrm{II}}(s; x) := \log\det(I - \mathbb{K}_s^{\mathrm{P}_{\mathrm{II}}^{\mathrm{k}}}).$$

Then, we have, as $s \to +\infty$,

$$F_{\mathrm{II}}(s; x) = -\frac{c_0^2}{2(2k+1)} s^{2(2k+1)} - \frac{c_0}{k+1} x s^{2(k+1)} - \frac{1}{2}(xs)^2 - \frac{2k+1}{4}\log s$$
$$+ \int_x^{+\infty} (\tau - x) q_{2k}^2(\tau) d\tau - \frac{1}{4}\log c_0 + \frac{1}{12}\log 2 + 3\zeta'(-1) + o(1),$$

with $c_0 = \frac{(2k)!}{k!^2}$.

The Painlevé II hierarchy

- The m-th member of $\mathrm{P}_{\mathrm{II}}^{\mathrm{m}}$ is a nonlinear ordinary differential equation of order $2m$ defined by

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}+2q\right)\mathcal{L}_m\left[q_x-q^2\right]+\sum_{i=1}^{m-1}\tau_i\left(\frac{\mathrm{d}}{\mathrm{d}x}+2q\right)\mathcal{L}_i\left[q_x-q^2\right]=xq-\alpha, \quad m\in\mathbb{N}^+,$$

  where the operator $\mathcal{L}$ satisfies the Lenard recursion relation

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{L}_{j+1}f=\left(\frac{\mathrm{d}^3}{\mathrm{d}x^3}+4f\frac{\mathrm{d}}{\mathrm{d}x}+2f_x\right)\mathcal{L}_jf, \quad \mathcal{L}_0f=\frac{1}{2}, \quad \mathcal{L}_j0=0, \quad j\geq 1.$$

- If $m=1$, one recovers the PII equation $y''(x)=xy(x)+2y^3(x)-\alpha$.
- If $m=2$, we have

$$q_{xxxx}-10qq_x^2-10q^2q_{xx}+6q^5+\tau_1(q_{xx}-2q^3)=xq-\alpha.$$

A special solution to the Painlevé II hierarchy

- For simplicity, we focus on the case $\alpha = 0$ and $\tau_k = 0, k = 1, \cdots, m-1$.
- For each positive integer $m \in \mathbb{N}^+$, there exists a real solution $q_m(x)$ such that Cafasso, Claeys and Girotti (IMRN, 2019)

$$q_m((-1)^{m+1}x) \sim \mathcal{O}\Big(e^{-Cs^{\frac{2m+1}{2m}}}\Big),\ x \to +\infty,\ \ q_m((-1)^{m+1}x) \sim \left(\frac{m!^2}{(2m)!}\right)^{\frac{1}{2m}} |x|^{\frac{1}{2m}},\ x \to -\infty.$$

- This solution is called the Hastings-McLeod solution of $\mathrm{P}_{\mathrm{II}}^{\mathrm{m}}$, which appears in our main theorem.

Higher order Airy-kernel determinant

- Our uniform asymptotic results reproduce the following large gap asymptotics previously established by Cafasso, Claeys and Girotti (IMRN, 2019)

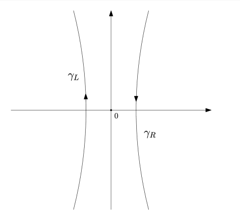
$$\ln \det(I - \mathbb{K}_x) = -\int_x^{+\infty} (\tau - x) q_{2k}^2(\tau) d\tau$$

$$= -c_1 |x|^{2+\frac{1}{k}} - \frac{2k+1}{24k} \log|x| + \frac{1}{24k} \log c_0 - \frac{1}{12} \log k + \zeta'(-1) + o(1)$$

with $c_0 = \frac{(2k)!}{k!^2}$ and $c_1 = \frac{k^2}{(k+1)(2k+1)} \left(\frac{(2k)!}{k!^2}\right)^{-\frac{1}{k}}$.

- $\mathbb{K}_x$ is the trace-class operator acting on $L^2(x, \infty)$ with the following higher order Airy kernel:

$$K(u, v) = \frac{1}{(2\pi i)^2} \int_{\gamma_R} dz \int_{\gamma_L} dw \frac{e^{\frac{(-1)^{k+1}}{2k+1}(z^{2k+1} - w^{2k+1}) - uz + vw}}{w - z}.$$



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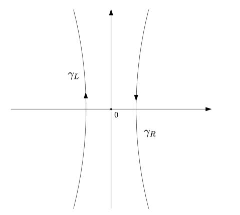

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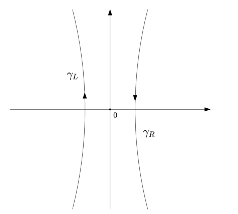

Relation with mKdV hierarchy

- We have

$$\frac{\partial^2}{\partial x^2} \log F_{\mathrm{II}}(s;x) = (q_m((-1)^{m+1}x))^2 - u(x;s)^2,$$

where q_m is the Hastings-McLeod solution of $\mathrm{P}_{\mathrm{II}}^{\mathrm{m}}$, and $u(x;s)$ is a special smooth solution to the coupled P_{II} hierarchy.

- Let

$$u(x;s) = sy\left((-1)^{m+1}xs, (2m+1)^{-1}s^{2m+1}\right),$$

we show that $y(x,t)$ is a solution of the mKdV hierarchy

$$y_t + \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} + 2y\right)\mathcal{L}_m[y_x - y^2] = 0, \qquad m \in \mathbb{N}^+.$$

- When $m = 1$, we have the modified Korteweg–de Vries (mKdV) equation

$$y_t - 6y^2 y_x + y_{xxx} = 0, \qquad x \in \mathbb{R}, \qquad t > 0.$$

- Therefore, for each $m \in \mathbb{N}^+$, we have found a Fredholm determinant solution to the mKdV hierarchy.

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Small-time asymptotics of the mKdV solution

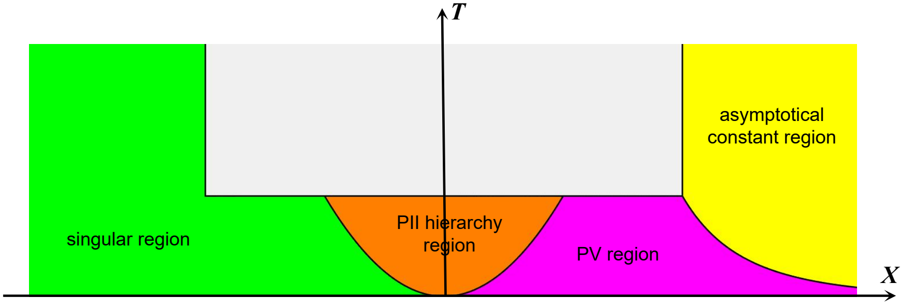

Figure: Asymptotic regions for $y((-1)^{m+1}X, T)$ on the (X, T) plane.

Small-time asymptotics of the mKdV solution

Theorem (D., Long, Xu, Yao and Zhang, in preparation)

With $c_0 = \frac{(2m)!}{(m!)^2}$ *and* $\epsilon_m \in (0, \frac{1}{2m+1})$. *For any* $m \in \mathbb{N}^+$, *there exist constants* $M, M' > 0$ *such that*

(i) *singular region: uniformly for* $X \le -MT^{\frac{1}{2m+1}-\epsilon_m}$,

$$y((-1)^{m+1}X, T) = \left(\frac{|X|}{c_0 T}\right)^{\frac{1}{2m}} + \mathcal{O}\left(|X|^{-2-\frac{1}{m}} T^{\frac{1}{m}}\right), \quad T \to 0^+;$$

(ii) $\mathrm{P_{II}}$ *hierarchy region: uniformly for* $|X| \le MT^{\frac{1}{2m+1}-\epsilon_m}$,

$$y((-1)^{m+1}X, T) = \frac{q_m\left(X((2m+1)T)^{-\frac{1}{2m+1}}\right)}{((2m+1)T)^{\frac{1}{2m+1}}} + \mathcal{O}\left(T^{\frac{1}{2m+1}}\right), \quad T \to 0^+,$$

where q_m *is the Hastings-McLeod solution of the* m*-th member of the* $\mathrm{P_{II}}$ *hierarchy;*

Small-time asymptotics of the mKdV solution

Theorem (Continued)

(iii) $\mathrm{P_V}$ *region: uniformly for* X *with* $MT^{\frac{1}{2m+1}-\epsilon_m} \leq X \leq M'T^{-\frac{1}{2m+1}}$,

$$y((-1)^{m+1}X, T) = \frac{(\sigma_V(X) - X\sigma_V'(X))^{\frac{1}{2}}}{X} + \mathcal{O}(T^{\frac{1}{2m}}X^{-1}), \quad T \to 0^+,$$

where σ_V *is the smooth solution of the* σ*-form* $\mathrm{P_V}$ *equation in* (8)*;*

(iv) *asymptotically constant region: uniformly for* $X \geq M'T^{-\frac{1}{2m+1}}$

$$y((-1)^{m+1}X, T) = 1 + \mathcal{O}\left(T^{\frac{2}{2m+1}}(X + c_0T)^{-2}\right), \quad T \to 0^+.$$

Main issue in determining the constant

- Due to the integrable structure of the $\mathrm{P}_{\mathrm{I}}^{\mathrm{k}}$ kernel, $\frac{\partial F}{\partial s}(s;x)$ is related to a Riemann-Hilbert (RH) problem.
- As $F(+\infty;x) = 0$, we have

$$F(s;x) = -\int_s^{+\infty} \frac{\partial F}{\partial \tau}(\tau;x)d\tau.$$

  By performing the powerful Deift-Zhou nonlinear steepest descent analysis for the associated RH problem, one obtains the asymptotics of $\frac{\partial F}{\partial s}(s;x)$ as $s \to -\infty$, which yields the large gap asymptotics of F except the constant term.

- The challenge to determine the constant term lies in the fact that one needs to understand detailed information of $\frac{\partial F}{\partial s}(s;x)$ across an infinite interval $(s,+\infty)$. This seems impracticable since the function $\frac{\partial F}{\partial s}(s;x)$ is highly transcendental. This limitation explains why most of the similar constant problems remain open.

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Ideas of our approach

- Our key idea is to investigate uniform asymptotics of the partial derivatives of $F(s;x)$ with respect to both s and x.
- The motivation follows from the following formula

$$F(s;x) = -\int_x^{x_0} \frac{\partial F}{\partial \mu}(s;\mu)d\mu - \int_s^{+\infty} \frac{\partial F}{\partial \tau}(\tau;x_0)d\tau, \tag{7}$$

which is valid for any real x_0.

- The arbitrariness of x_0 provides the flexibility of choosing the variable $|x_0|$ to be large alongside the variable $|s|$. This is essential in the sense that the behaviors of $\frac{\partial F}{\partial x}(s;x)$ and $\frac{\partial F}{\partial s}(s;x)$ degenerate in the asymptotic regime, which can be readily established through their connections with RH problems.

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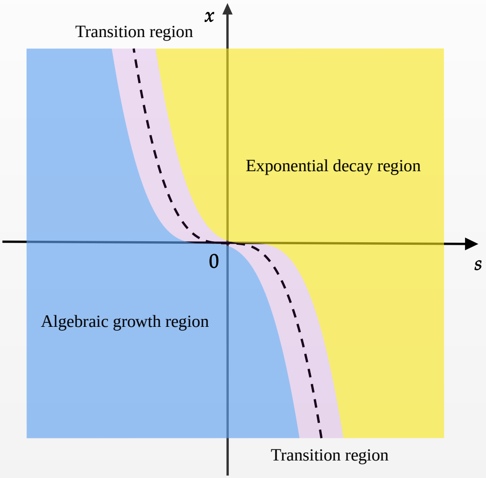
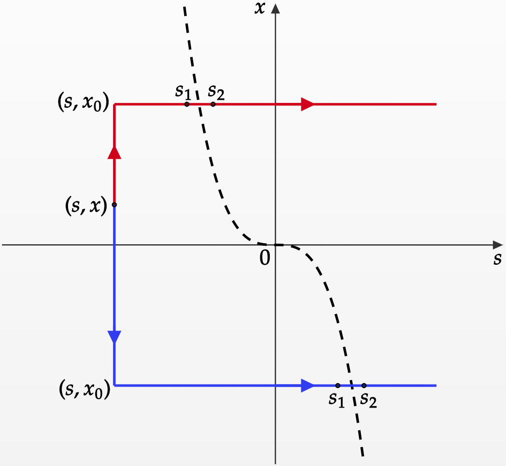
Three asymptotic regions

Both $\frac{\partial F}{\partial s}(s;x)$ and $\frac{\partial F}{\partial x}(s;x)$ exhibit qualitatively different asymptotic behaviors in three different regions of the (x, s)-plane.



The integration contour

We will set $x_0 = \pm|s|^{2k+1}$ in (7), replace the contour of integration therein by the lines depicted in Figure below.



Summary

- We present a novel approach to establish the multiplicative constant in the large gap asymptotics of unitary random matrix ensembles near critical points.
- The constant term involves of an integral of particular real and pole-free solutions for $\mathrm{P}_{\mathrm{I}}^{2\mathrm{k}}$ or $\mathrm{P}_{\mathrm{II}}^{2\mathrm{k}}$.
- We have found a special solution of the mKdV hierarchy expressed in terms of the Fredholm determinant associated with the $\mathrm{P}_{\mathrm{II}}^{\mathrm{k}}$ kernel. The small time asymptotics of the solution is derived on the whole real line.
- Our analysis employs a delicate uniform steepest descent method for RH problems. This approach is expected to be applied to for determine critical constants in similar problems from mathematical physics.

Thank You!

&

Happy Birthday, Peter!