Recent progress on free energy expansions of two-dimensional Coulomb gases

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Log-gases in Caeli Australi:

Recent Developments in and Around Random Matrix Theory,

August 5, 2025

Outline

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Complex Ginibre Matrix

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$$G = (G_{jk})_{j,k=1}^{N}$$

where

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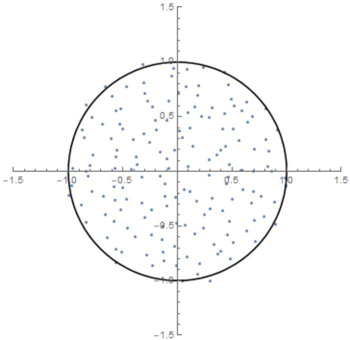

Eigenvalues of $\boldsymbol{G}$ $(N = 160)$

The Circular Law

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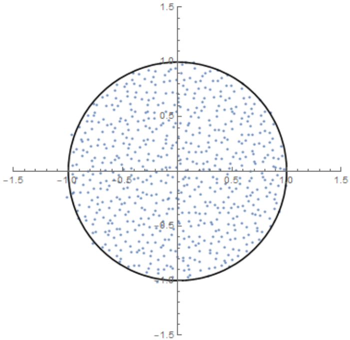

Eigenvalues of $\boldsymbol{G}$ $(N = 640)$

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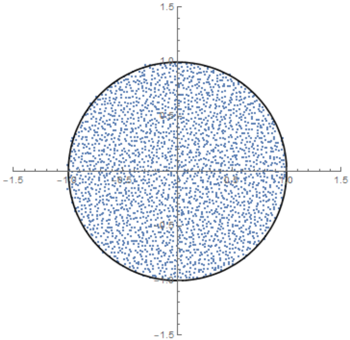

Eigenvalues of $\boldsymbol{G}$ $\color{red}{(N = 2560)}$

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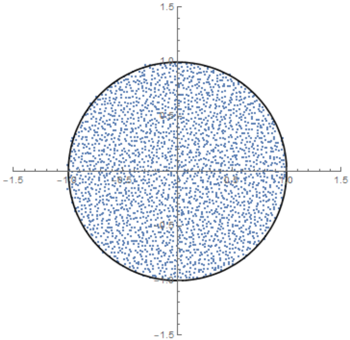
- **Joint PDF** for eigenvalues $\boldsymbol{z} = \{z_j\}_{j=1}^{N}$:

$$\frac{1}{Z_N^{\mathrm{cGin}}} \prod_{j>k=1}^{N} |z_j - z_k|^2 e^{-N\sum_{j=1}^{N} |z_j|^2},$$

where

$$Z_N^{\mathrm{cGin}} = \frac{N!}{N^{N(N+1)/2}} \prod_{j=1}^{N-1} j!$$



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2D Coulomb Gas Ensemble

- **2D Coulomb Gas**: the system $\boldsymbol{z} = \{z_j\}_{j=1}^N \in \mathbb{C}^N$ with

$$\frac{1}{Z_{N,Q}^{(\beta)}} \prod_{j>k=1}^{N} |z_j - z_k|^{\beta} e^{-\frac{\beta N}{2} \sum_{j=1}^{N} Q(z_j)}$$

$$= \frac{1}{Z_{N,Q}^{(\beta)}} e^{-\frac{\beta N^2}{2} H_N(\boldsymbol{z})}, \qquad H_N(\boldsymbol{z}) = \frac{1}{N^2} \sum_{j \neq k} \log \frac{1}{|z_j - z_k|} + \frac{1}{N} \sum_{j=1}^{N} Q(z_j)$$

where $Q : \mathbb{C} \to \mathbb{R}$ which satisfies $Q(z) \gg \log |z|$ near infinity.

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- **Equilibrium Convergence** (Johansson '98):

$$\frac{1}{N}\sum_{j=1}^N \delta_{z_j}(z) \longrightarrow d\mu_Q(z)$$

where μ_Q is a unique minimiser of the energy

$$I_Q[\mu] = \int_{\mathbb{C}^2} \log\frac{1}{|z-w|}\, d\mu(z)\, d\mu(w) + \int_{\mathbb{C}} Q\, d\mu.$$

Logarithmic Potential Theory: the Droplet

- **The Laplacian Growth**: μ_Q is of form

$$d\mu_Q(z) = \Delta Q(z) \cdot \mathbb{1}_S(z) \, \frac{d^2 z}{\pi}, \qquad \Delta = \partial\bar{\partial}$$

where S is called the *droplet*.

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- **Geometries of the Droplets** (for Hele-Shaw type potentials $\Delta Q = \text{const}$):

Sakai: *Regularity of a boundary having a Schwarz function,* Acta Math. **166** (1991), 263–297.

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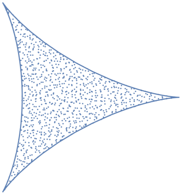
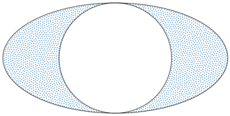
- **Construction of the Droplets** (with singular boundary points):

  $\ast$ *Schwarz Function Theory & Conformal Analysis of Hele-Shaw flow*



Bleher-Kuijlaars '12

Akemann-B.-Kang '21



B.–Yoo '24

Partition Functions

- **Partition Functions**:

$$Z_{N,Q}^{(\beta)} = \int_{\mathbb{C}^N} \prod_{j>k=1}^{N} |z_j - z_k|^\beta \prod_{j=1}^{N} e^{-\frac{\beta N}{2} Q(z_j)} \frac{d^2 z_j}{\pi}$$

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- **Large-N Expansion**:

$$\log Z_{N,Q}^{(\beta)} \sim -\frac{\beta}{2} I_Q[\mu_Q]\, N^2$$

Johansson, *On fluctuations of eigenvalues of random Hermitian matrices*, Duke Math. J. **91** (1998), 151–204.

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Sandier-Serfaty, *2D Coulomb gases and the renormalized energy*, Ann. Probab. **43** (2015), 2026–2083.

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$$\log Z_{N,Q}^{(\beta)} \sim -\frac{\beta}{2} I_Q[\mu_Q]\, N^2 + \frac{\beta}{4} N \log N - \Big(C(\beta) + \Big(1 - \frac{\beta}{4}\Big) E_Q[\mu_Q] \Big) N$$

where $C(\beta)$ is a constant independent of the potential Q and

$$E_Q[\mu_Q] = \int_{\mathbb{C}} \log(\Delta Q)\, d\mu_Q$$

Leblé-Serfaty, *Large deviation principle for empirical fields of log and Riesz gases*, Invent. Math. **210** (2017), 645–757.

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cf. Quantitative error bounds: Bauerschmidt-Bourgade-Nikula-Yau '19, Armstrong-Serfaty '21, Serfaty '23

Partition Functions: Predictions from Conformal Field Theory

$$\log Z_{N,Q}^{(\beta)} \sim C_0 N^2 + C_1 N \log N + C_2 N + C_3 \sqrt{N} + C_4 \log N + C_5$$

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- **Surface Tension** (Lutsyshin & Can-Forrester-Téllez-Wiegmann '15):

$$C_3 = (\text{\# of components of } \partial S_Q) \cdot \frac{4}{3\sqrt{\pi}} \log(\beta/2)$$

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- **Zeta-regularised Determinant of Laplacian** (Zabrodin-Wiegmann '06):

$$C_5 = -\frac{1}{2}\log\left(\frac{\det_\zeta(-\Delta_{\mathbb{C}\setminus S_Q})}{\det_\zeta(-\Delta_{\mathbb{C}})}\right) + \mathfrak{c}(\beta) + \mu(\beta)\oint_{\partial S_Q} \partial_n \phi \, ds$$
$$+ \frac{(\beta-4)^2}{16\beta}\left(\int_{\mathbb{C}} |\nabla\phi|^2 - \mathbb{1}_{S_Q^c}|\nabla(\phi - \phi^{\mathrm{H}})|^2 \frac{d^2 z}{\pi}\right),$$

where $\phi = \frac{1}{2}\log \Delta Q$ and some unknown constants $c(\beta), \mu(\beta)$.

Applications of Free Energy Expansions

- Law of large numbers and fluctuation theory for Coulomb gases
- Geometric properties of limiting droplets
- Large deviation probabilities, e.g. hole probabilities
- Log-correlated fields and Gaussian multiplicative chaos
- Large deviation principles in integrable models, e.g. last passage percolation

Outline

Partition Functions of Determinantal Coulomb Gases

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$$Z_{N,Q} \equiv Z_{N,Q}^{(2)} = \int_{\mathbb{C}^N} \prod_{j>k=1}^{N} |z_j - z_k|^2 \prod_{j=1}^{N} e^{-NQ(z_j)} \frac{d^2 z_j}{\pi}$$

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- **Vandermonde Identity**: for any monic polynomial P_j of degree j,

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- **Planar Orthogonal Polynomial**: monic P_j's such that

$$\int_{\mathbb{C}} P_j(z) \overline{P_k(z)} e^{-NQ(z)} \frac{d^2 z_j}{\pi} = h_j \, \delta_{jk}.$$

cf. (Hedenmalm-Wennman '21)

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- **Partition Function and Orthogonal Norm**:

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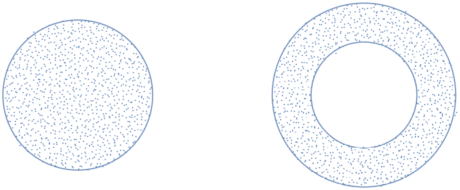

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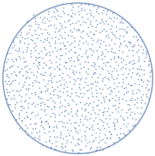

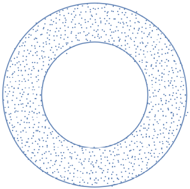

cf. Under hard edge constraints (with applications to hole probabilities)
(Allard-Forrester-Lahiry-Shen '25, Charlier-Noda '25+)

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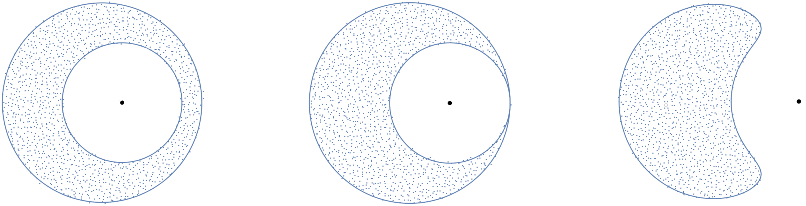

$c < c_{\text{cri}}$ $\quad$ $c = c_{\text{cri}}$ $\quad$ $c > c_{\text{cri}}$

Balogh-Bertola-Lee-McLaughlin, *Strong asymptotics of the orthogonal polynomials with respect to a measure supported on the plane,* Comm. Pure Appl. Math. **68** (2015), 112–172.

- **Free Energy Expansion for the Conditional Ginibre Ensemble**:

$$\log Z_N(a,c) = -I_Q[\sigma_Q]N^2 + \frac{1}{2}N\log N + \Big(\frac{\log(2\pi)}{2} - 1\Big)N + \frac{6-\chi}{12}\log N + \frac{\log(2\pi)}{2} + \chi\,\zeta'(-1) + \mathcal{F}(a,c) + O(\frac{1}{N})$$

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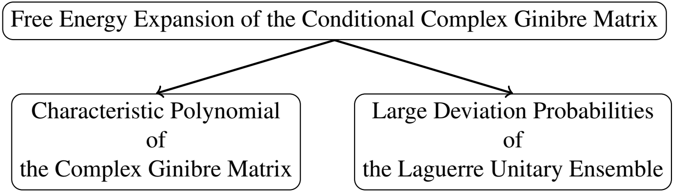
- **Applications**: from the duality (Nishigaki-Kamenev '02, Forrester-Rains '09, Forrester '25)



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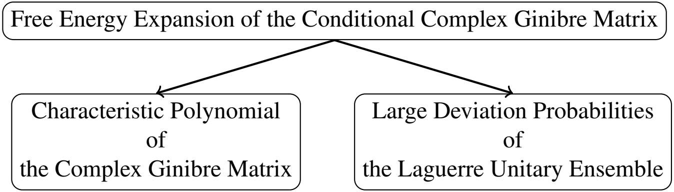

cf. Large deviation probabilities in 1D: (Ben Arous-Dembo-Guionnet '01, Dean-Majumdar '06, '08, Vivo-Majumdar-Bohigas '07, Katzav-Castillo '10, Majumdar-Schehr '14, Perret-Schehr '16)

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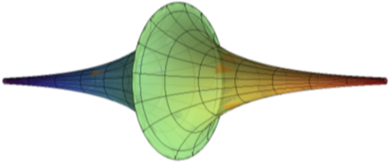

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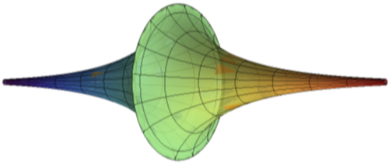

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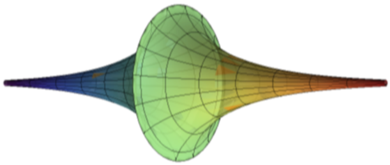

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- **Lemniscate Potential** (Balogh-Grava-Merzi '17, Bertola-Elias Rebelo-Grava '18) :

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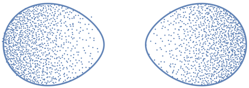
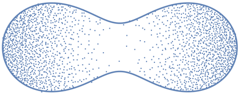


Multi-component: $t > t_{\text{cri}}$



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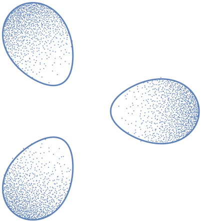

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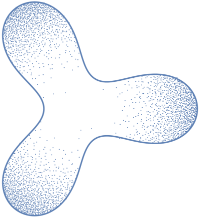

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- **Anomalous Free Energy Expansions**:

$$\log Z_{N,Q} = -I_Q[\mu_Q]N^2 + \frac{1}{2}N\log N + \left(\frac{\log(2\pi)}{2} - 1 - \frac{1}{2}E_Q[\mu_Q]\right)N + \frac{6-\chi}{12}\log N + \chi\,\zeta'(-1) + \frac{\log(2\pi)}{2} + \mathcal{F}[Q] + \mathcal{G}_N[Q] + \mathcal{H}_N[Q] + o(1)$$

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- **Conformal Singularity Regime** $\mathcal{H}_N[Q]$
  - Divergent spectral determinant
  - Additional logarithmic growth depending on the singularity structure

IV: Lemniscate Ensembles (Anomalous Free Energy Expansions)

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For the lemniscate ensembles, the free energy expansion holds, where $\mathcal{F}$, $\mathcal{G}_N$ and $\mathcal{H}_N$ are given as follows.

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  Here, G is the Barnes G-function.

Remarks

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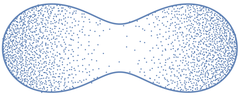
  with explicit $\mathcal{C}_m$ in terms of Bernoulli numbers.

  cf. quantitative error terms of (Webb-Wong '19) (Deaño-McLaughlin-Molag-Simm '25)

IV: Lemniscate Ensembles (Anomalous Free Energy Expansions)

- **Logarithmic Divergence in $\mathcal{H}_N$:**

$$\mathcal{H}_N[W] = \frac{(d-1)^2}{12d} \log N + O(1)$$



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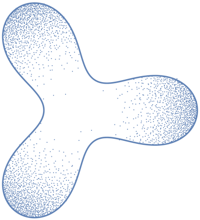

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- **Geometric Functional in the Spectral Determinant**:

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- **Extended Conjecture of Jancovici et al.**: If there exists $p \in S_Q$ such that the density behaves as

$$O(|z-p|^{2d-2}) \quad \text{or} \quad O(|z-p|^{2/d-2})$$

for some $d \in \mathbb{N}$, then the coefficient of the $\log N$ term in the expansion is

$$\frac{6-\chi}{12} + \frac{(d-1)^2}{12d},$$

where χ denotes the Euler characteristic of the droplet.

Recent Progress of Free Energy Expansions of Determinantal Coulomb Gases

- **Radially Symmetric Ensembles**
- **Conditional Ginibre Ensembles**
  - extremal eigenvalues of the LUE
- **Conditional Truncated Unitary Ensembles**
  - extremal eigenvalues of the JUE
  - last passage time of the geometric last passage percolation
- **Lemniscate Ensembles** *(Anomalous Free Energy Expansions)*
  - multi-component: oscillatory behaviour
  - conformal singularity: divergent spectral determinant & beyond Jancovici et al.

Summary

- **Partition Functions**:

$$Z_{N,Q}^{(\beta)} = \int_{\mathbb{C}^N} \prod_{j>k=1}^{N} |z_j - z_k|^\beta \prod_{j=1}^{N} e^{-\frac{\beta N}{2} Q(z_j)} \frac{d^2 z_j}{\pi}$$

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$$\log Z_{N,Q}^{(\beta)} \sim C_0 N^2 + C_1 N \log N + C_2 N + C_3 \sqrt{N} + C_4 \log N + C_5$$

with *potential theoretic/topological/conformal geometric* information.
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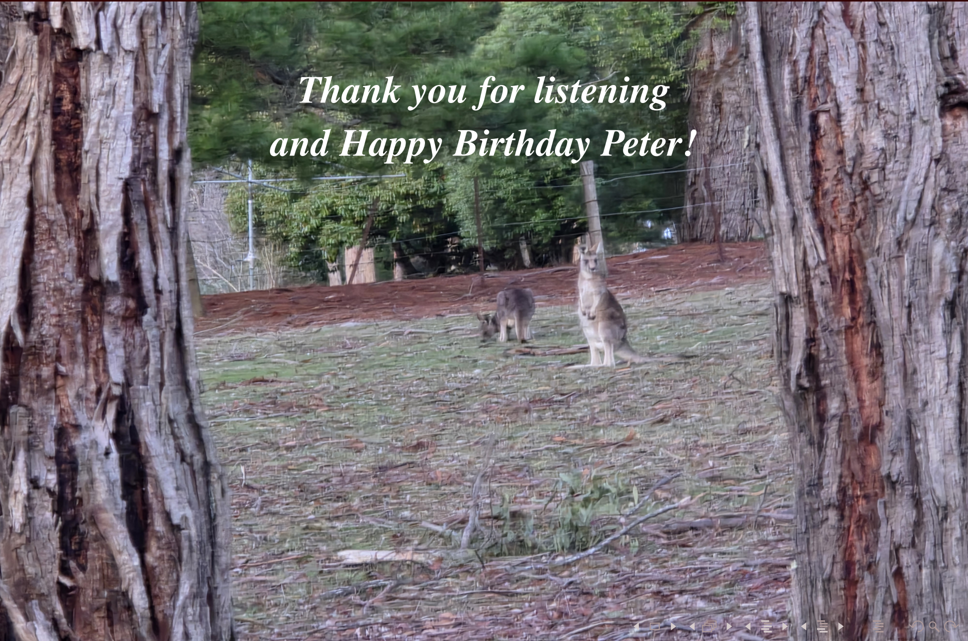
- **Determinantal Coulomb Gases**: for $\beta = 2$, the determinantal structure, together with techniques from *orthogonal polynomials* and *duality identities*, enables the verification of the conjecture for certain classes of potentials and leads to several applications.


Thank you for listening
and Happy Birthday Peter!