Fredholm Determinants and Painlevé Transcendents

a pragmatist's perspective on integrability

```
airy := [ q[s] = -q*u + p, p[s] = s*q + p*u - 2*q*v, u[s] = -q^2, v[s] = -p*q ]:

vars := [ s, q, p, u, v ]:
R := DifferentialRing(derivations = [s], blocks = [ [v,p,u], [q] ]):
Ids := RosenfeldGroebner(airy, R):
FI := findAllFirstIntegralsLU(Ids[1], newMonomialIterator(vars, 3), R, {}):
Ids := RosenfeldGroebner([op(airy), op(FI[2..-1])], R):
Equations(Ids[1])[-1];

> -2*q^3  - s*q + q[s, s]
```

Tracy–Widom in the 21^{st} century: from Airy kernel encoding to Painlevé II

H. Flaschka's definition of integrability

H. McKean '03: the only honest one around

P. Deift '19: the 'wild west' spirit

'You didn't think I could integrate that, but I can!'

tools

- isomonodromic deformations
- Riemann–Hilbert problems
- inverse scattering
- Hirota bilinear forms
- τ-functions
- differential algebra

solutions

- classical special functions (linear ODEs)
- linear integral equations
- finite and infinite determinants
- nonlinear ODEs w/ Painlevé property ‡
  - elliptic functions (1^{st} order)
  - Painlevé transcendents (2^{nd} order)
  - ?

‡) general solution is single–valued in its domain of definition

Ivar Fredholm (1866–1927)

determinant of integral operator (1899)

$$Ku(x) = \int_a^b K(x,y)u(y)\,dy$$

$$\det(I + zK) = \sum_{m=0}^{\infty} \frac{z^m}{m!} \int_{[a,b]^m} \det_{i,j=1}^{m} K(t_i,t_j)\,dt$$

Paul Painlevé (1863–1933)

six families of irreducible‡ transcendental functions (1895)

classifying $y'' = F(x,y,y')$ w/ Painlevé property

PI: $y'' = 6y^2 + x$

PII: $y'' = 2y^3 + xy + \alpha$

PIII: $y'' = y^{-1}y'^2 - x^{-1}y' + x^{-1}(\alpha y^2 + \beta) + \gamma y^3 + \delta y^{-1}$

PIV: $y'' = (2y)^{-1}y'^2 + 3y^3/2 + 4xy^2 + 2(x^2 - \alpha)y + \beta y^{-1}$

PV: $y'' = (3y-1)(2y(y-1))^{-1}y'^2 - x^{-1}y' + \gamma x^{-1}y + (y-1)^2x^{-2}(\alpha y + \beta y^{-1}) + \delta y(y+1)(y-1)^{-1}$

PVI: $y'' = (y^{-1} + (y-1)^{-1} + (y-x)^{-1})y'^2/2 - (x^{-1} + (x-1)^{-1} + (y-x)^{-1})y' + \dfrac{y(y-1)(y-x)}{x^2(x-1)^2}\left(\alpha + \dfrac{\beta x}{y^2} + \dfrac{\gamma(x-1)}{(y-1)^2} + \dfrac{\delta x(x-1)}{(y-x)^2}\right)$

‡) Painlevé 1903: refers to Drach's (still unfinished) *infinite* dimensional differential Galois theory

Nishioka '87, Umemura '87: proof by Kolchin's *finite* . . . .

soft-edge scaling limit$|_{\beta=2}$

$$F(s;\xi) = \sum_{n=0}^{\infty} \mathbb{P}(n = \#\{\text{levels} > s\}) \cdot (1-\xi)^n$$

Forrester '93

$$F(s;\xi) = \det\left(I - K|_{L^2(s,\infty)}\right)$$

w/ kernel

$$K(x,y) = \frac{A(x)A'(y) - A'(x)A(y)}{x-y}$$

Recently, a numerical analyst [B.] has shown that the most efficient way‡ *to compute spacing distributions in classical RMT is to use Fredholm determinant formulas.* — Forrester '10

Tracy–Widom '94

$$F(s;\xi) = \exp\left(-\int_s^\infty (x-s)q(x)^2\,dx\right)$$

w/ Painlevé II

$$q'' = xq + 2q^3$$

$$q(x) \simeq A(x) \qquad (x \to \infty)$$

Without the Painlevé representations, the numerical evaluation of the Fredholm determinants is quite involved. — Tracy–Widom '00

$$A(x) = \sqrt{\xi}\,\mathrm{Ai}(x)$$

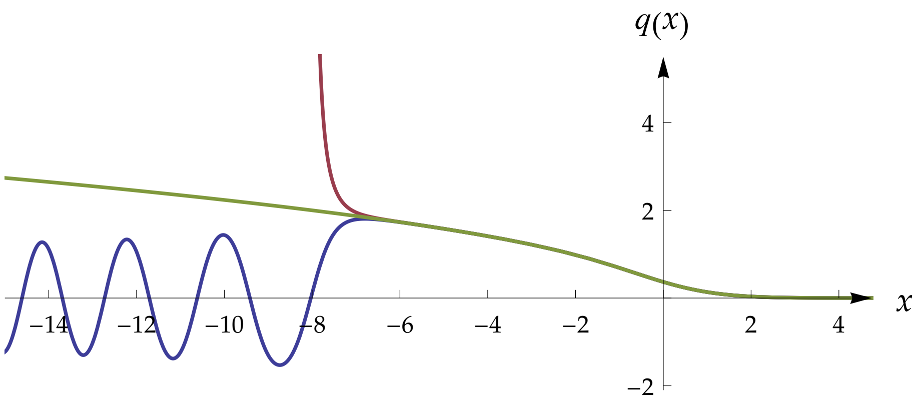
‡) basically, by proving the convergence of Nyström's method

instability

solution of Painlevé II,

$$q'' = xq + 2q^3, \qquad q(x) \simeq \sqrt{\xi}\,\mathrm{Ai}(x) \quad (x \to \infty),$$

separatrix for $\xi = 1$ $\rightsquigarrow$ backwards IVP highly unstable



$q(x)$ with $\sqrt{\xi} = 1 - 10^{-8}, 1, 1 + 10^{-8}$

$q(x)$ with $\sqrt{\xi} = 1 - 10^{-16}, 1, 1 + 10^{-16}$

consequence

- calculate q via a BVP solution $\rightsquigarrow$ *connection formula* Hastings–McLeod '80

$$q(x) \simeq \mathrm{Ai}(x) \quad (x \to \infty) \qquad \Rightarrow \qquad q(x) \simeq \sqrt{-x/2} \quad (x \to -\infty)$$

linear integral equations

$$\sigma(s) := \partial_s \log\det(I - K_s), \quad Q(s,t) := (I-K_s)^{-1}A(t), \quad K_s := K|_{L^2(s,\infty)},$$

$$(\partial_x + \partial_y)K \overset{\text{Airy}}{\underset{\text{diff.eq.}}{=}} -A\otimes A \rightsquigarrow \sigma(s) = \operatorname{tr}\big((I-K_s)^{-1}A\otimes A\big) \rightsquigarrow \sigma'(s) = -Q(s,s)^2$$

Ablowitz–Segur '77 $0 \leqslant \xi < 1$; Hastings–McLeod '80: $\xi = 1$

- Neumann series of $(I-K_s)^{-1}$
- apply termwise $L = (\partial_s + \partial_t)^2 - t$
- repeated integration by parts

$$\rightsquigarrow \boxed{LQ(s,t) = 2Q(s,s)^2 Q(s,t)} \rightsquigarrow Q(s,s) \text{ solves Painlevé II}$$

$Q(s,s) \sim A(s)$

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the fact can also be obtained by combining results in [AS77,HM80] — Tracy/Widom '94

the old and cumbersome method — Mehta '04

linear integral equations algorithmic (Tracy–Widom '94)

$$\begin{matrix} q \\ p \end{matrix} := (I - K_s)^{-1} \begin{matrix} A \\ A' \end{matrix}(s), \quad \Bigg| \quad \begin{matrix} u \\ v \end{matrix} := \langle (I - K_s)^{-1} A, \begin{matrix} A \\ A' \end{matrix} \rangle \quad \Bigg| \quad \sigma = u$$

$\rightsquigarrow$ polynomial ODEs, 'encoding' Airy kernel

$$q' = p - qu, \quad p' = sq + pu - 2qv, \quad u' = -q^2, \quad v' = -pq$$

first integrals differential algebra (Boulier–Lamaire '15)

$$q^2 - u^2 + 2v \overset{s \to \infty}{=} 0, \quad 2pqu - p^2 - 2q^2 v + sq^2 + u \overset{s \to \infty}{=} 0$$

integration differential algebra: Rosenfeld–Gröbner elimination (Boulier et al. '95)

- elimination order: $\ldots, q, \sigma$

$$\boxed{\sigma''^2 + 4\sigma'(\sigma'^2 - s\sigma' + \sigma) = 0}$$

$$q^2 = -\sigma'$$

- σ-PII$\big|_{\alpha = -\frac{1}{2}}$

- elimination order: $\ldots, \sigma, q$

$$\boxed{q'' = sq + 2q^3}$$

$$\sigma = q'^2 - sq^2 - q^4$$

- PII$\big|_{\alpha=0}$

despite several efforts with the help of prominent persons, including the four authors, we never understood the original [isomonodromic] proof by Jimbo–Miwa–Môri–Sato '80 — Mehta '92

our proof is a simplification of Mehta's simplified proof — Tracy–Widom '92

$$\sigma := -s\partial_s \log\det(I - K_s), \quad K_s := K|_{L^2(-\frac{1}{2}s, \frac{1}{2}s)} \text{ sine kernel } \Big|_{\text{no } \pi}$$

linear integral equations algorithmic

$\rightsquigarrow$ encoding polynomial ODEs with $\sigma = s(p^2 + q^2) - 4p^2q^2$

$$sq' = sp/2 - 2pq^2, \quad sp' = -sq/2 + 2p^2 q$$

→ no first integrals here ←

integration differential algebra

elimination order: $\ldots, \sigma$

$$\boxed{(s\sigma'')^2 + 4(s\sigma' - \sigma)(s\sigma' - \sigma - \sigma'^2) = 0} \qquad \sigma\text{-PV}\big|_{t \mapsto -2t, \vec{v}=0}$$

Tracy–Widom '92 : complexify $r = p + iq$ $\rightsquigarrow$ 'Gaudin relations' $\overset{\text{Mehta '92}}{\rightsquigarrow}$ σ-PV

$$\sigma := -s\partial_s \log\det(I - K_s), \quad K_s := K|_{L^2(0,s)}$$ Bessel kernel

linear integral equations algorithmic; Tracy–Widom '94

$\rightsquigarrow$ encoding polynomial ODEs with $\sigma = u/4$

$$sq' = p + qu/4, \quad sp' = (a^2 - s)q/4 - pu/4 + qv/2, \quad u' = q^2, \quad v' = pq$$

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first integrals differential algebra

$$4sq^2 - 4u - u^2 - 8v \overset{s \to 0}{=} 0, \quad 4a^2q^2 - 8pqu - 16p^2 + 8q^2v - u^2 - 8v \overset{s \to 0}{=} 0$$

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integration differential algebra

elimination order: $\ldots, \sigma$

$$\boxed{(s\sigma'')^2 + (4\sigma' - 1)(\sigma - s\sigma')\sigma' - a^2\sigma'^2 = 0}$$ σ-PIII$|_{v_1=v_2=a}$

Tracy–Widom '94:
- 2nd-order ODE for q, $q(x^2) = \frac{1+y(x)}{1-y(x)} \rightsquigarrow y$: PV$|_{\alpha=-\beta=a,\gamma=0,\delta=1/4} \overset{\text{alg. Bäcklund}}{\underset{\text{transform}}{\rightsquigarrow}}$ PIII
- '*sometimes convenient*': $q = \cos\psi \rightsquigarrow \sigma = s^2\psi''^2 + \frac{1}{4}s\cos^2\psi - a^2\cot^2\psi \rightsquigarrow \sigma$: σ-PIII

linear integral equations Neumann series

$$\partial_s \log\det(I - K_s) = -K(s,s) - (K^2)(s,s) - (K^3)(s,s) - \cdots$$

Bessel kernel: $4K(4x,4y) = \dfrac{(xy)^{a/2}}{\Gamma(a+1)\Gamma(a+2)}(1 + O(x) + O(y)) \rightsquigarrow \sigma(s) \sim \dfrac{1}{\Gamma(a+1)\Gamma(a+2)}\left(\dfrac{s}{4}\right)^{a+1}$

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more terms differential algebra: Rosenfeld–Gröbner elimination $\rightsquigarrow$ Taylor expansions (Boulier et al. '09)

$a = n \in \mathbb{Z}_{>0}$

$$\boxed{\sigma(s) = \frac{s^{n+1}}{4^{n+1} n!(n+1)!} + c_{n+2}s^{n+2} + c_{n+3}s^{n+3} + \cdots + c_m s^m + O(s^{m+1})}$$

$c_k \in \mathbb{Q}$: combinatorial meaning

- σ-PIII (cubic in σ) $\rightsquigarrow$ $c_{k+1} = \cdots \sum\sum \cdots |_{c_0,\ldots,c_k}$ $\rightsquigarrow$ $O(m^3)$ complexity ‡
- $\partial_s(\sigma\text{-PIII})/\sigma''$ $\rightsquigarrow$ Chazy-I

$$2s^2\sigma''' + 2s\sigma'' - 12s\sigma'^2 + 8\sigma\sigma' + 2(s - n^2)\sigma' - \sigma = 0$$

quadratic in σ $\rightsquigarrow$ $c_{k+1} = \cdots \sum \cdots |_{c_0,\ldots,c_k}$ $\rightsquigarrow$ $O(m^2)$ complexity ‡

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‡) m large: use floats w/ fixed high precision and rational reconstruction

observation

'$\doteq$' means '*equal up to elementary factors and shifts*'

$$\sigma \doteq \underbrace{F'/F}_{\text{meromorphic}}, \quad F = \underbrace{\det(I - K_s)}_{\text{holomorphic}}$$

long open problem solved by Nevanlinna theory (Hinkkanen–Laine, Shimomoura, Steinmetz '04)

$\rightarrow$ Painlevé transcendents are meromorphic $\leftarrow$

one strategy Painlevé 1910, Malmquist 1922, ..., Jimbo–Miwa '81, Okamoto '81

limited parameter space

PI–PVI have Hamiltonian structure $H \in \mathbb{Q}[s, q, p]$ (not unique)

PJ for short

$$q' \doteq \partial_p H, \quad p' \doteq -\partial_q H$$

- $H|_{\text{solution}} = \partial_s \log \tau = \tau'/\tau$ with τ holomorphic
- $\sigma \doteq H$ satisfies σ-PJ (w/ choices of convenience) $\rightsquigarrow$ σ, q meromorphic

consequence $\det(I - K_s) \doteq \tau$

Its et al. '90

K integrable kernel $\rightsquigarrow$ Jimbo–Miwa–Ueno '81 isomonodromic τ-fct of Schlesinger system: $\det(I - K) = \tau_{\text{JMU}}$

Hirota bilinear forms

$$D_s f \cdot g := \lim_{t\to s} \frac{f(t)g(s) - f(s)g(t)}{t-s} = \begin{vmatrix} f' & g' \\ f & g \end{vmatrix}$$

example: PII

$$(D_s^4 + 2sD_s^2)\tau \cdot \tau = \partial_s \tau^2$$

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extensive ⇝ generalization

advanced theory of τ-functions

Sato ’80, Segal–Wilson ’85

- Hirota bilinear forms ⇜⇝ ‘Plücker relations’
- τ-functions = determinants on infinite-dimensional Grassmannians

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general solution of PVI satisfies

Gavrylenko–Lisovyy ’18

$$\tau_{\text{VI}} \doteq \det\big(I - K|_{L^2(S^1)}\big), \quad K\text{: matrix kernel in terms of } {}_2F_1$$

PV/PIII: Cafasso–Gavrylenko–Lisovyy ’19, PII: Desiraju ’19

2D semi-infinite Toda lattice equation

$\tau_n = \tau_n(s,t)$, $\tau_0 \equiv 1$

$$\tau_n^2\, \partial_s \partial_t \log \tau_n = \boxed{\tfrac{1}{2} D_s D_t \tau_n \cdot \tau_n = \tau_{n-1}\tau_{n+1}}$$

explicit solution

Darboux 1887: studying Laplace invariants of surfaces with negative curvature

$$\tau_n = \det_{j,k=0}^{n-1} \partial_s^j \partial_t^k \tau_1$$

observe: τ_1 holomorphic $\rightsquigarrow$ τ_n holomorphic

unitary group integral

Rains '98: $\tau_n(x)$ is the generating function for the increasing subsequence problem

$$\tau_n(x) = \mathbb{E}_{M\in U(n)} e^{\sqrt{x}\,\mathrm{tr}(M+M^{-1})}, \quad \tau_1(x) = I_0(2\sqrt{x})$$

satisfies

Kharchev–Mironov '92

$$\tfrac{1}{2} D_s D_t \tau_n(st)\cdot\tau_n(st)\Big|_{s=t=\sqrt{x}} = \tau_{n-1}(x)\tau_{n+1}(x) \quad \rightsquigarrow \quad \tau_n(x) = \det_{j,k=0}^{n-1} \underbrace{\partial_s^j \partial_t^k I_0(2\sqrt{st})\Big|_{s=t=\sqrt{x}}}_{=I_{j-k}(2\sqrt{x})}$$

a powerful tool

birational maps $q \leftrightarrow q^*$ in the solution space of PJ s.t. Noumi–Yamada '98, Watanabe '98

- Hamiltonian structure preserved
- induces a shift $T\vec{v}$ in the parameter space $\vec{v}$ of the σ-PJ form
  affine Weyl group
- $\tau = \tau|_{\vec{v}} \leftrightarrow \tau^* = \tau|_{T\vec{v}}$

and Kajiwara et al. '01, Forrester–Witte '01–'04

- if v_0 s.t. $\tau_0 \equiv 1$
- then $\tilde{\tau}_n \doteq \tau_n$ satisfies Toda equation $\rightsquigarrow$ $\tau_n \doteq n$-dim determinant

Painlevé system	ODE for τ_1 (hypergeometric)	ON-polynomials
PII	Airy ${}_0F_1$	—
PIII	Bessel ${}_0F_1$	—
PIV	Weber ${}_1F_1$	Hermite
PV	Kummer ${}_1F_1$	Laguerre
PVI	Gauss ${}_2F_1$	Jacobi

finite matrix ensembles Tracy–Widom '94, Forrester–Witte '01–'04

ξ enters the boundary conditions

$$\det\left(I - \xi K_n^{\mathsf{GUE}}\big|_{L^2(s,\infty)}\right) = \tau_{\mathsf{IV}}(s)\big|_n$$

$$\det\left(I - \xi K_n^{\mathsf{LUE}_a}\big|_{L^2(0,s)}\right) = s^{an+n^2}\tau_{\mathsf{V}}(s)\big|_{n,a}$$

$$\det\left(I - \xi K_n^{\mathsf{JUE}_{a,b}}\big|_{L^2(0,s)}\right) = \tau_{\mathsf{VI}}(s)\big|_{n,a,b}$$

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identify unitary group integral τ_n

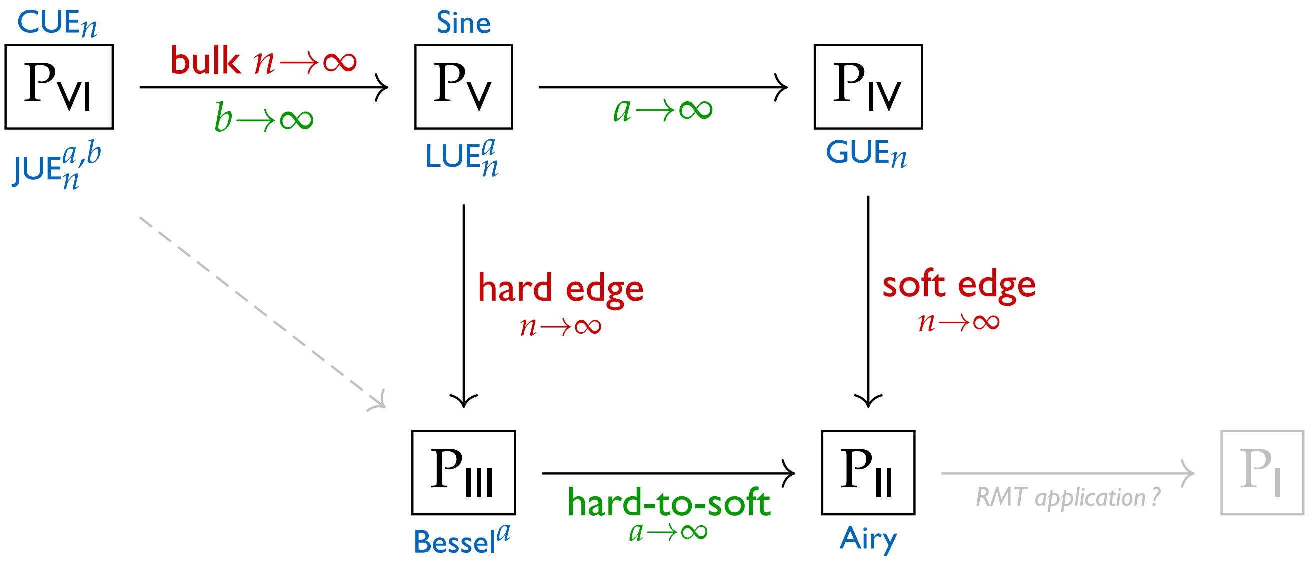
- τ_n same Toda equation as τ_{III} under canonical Bäcklund $(v_1, v_2) \mapsto (v_1+1, v_2+1)$
- $\tau_1 = \tau_{\mathsf{III}}\big|_{v_1=v_2=1} \rightsquigarrow \tau_n = \tau_{\mathsf{III}}\big|_{v_1=v_2=n}$
- unique boundary condition compatible with Toda no combinatorics needed

$$\sigma_n(4x) = x - x\partial_x \log \tau_n(x) = \frac{x^{n+1}}{n!(n+1)!} + O(x^{n+2}) \quad (x \to 0)$$

- compare with σ-PIII$\big|_{v_1=v_2=a}$-representation of Bessel kernel determinant

$$\det\left(I - K^{\mathsf{Bessel}}\big|_{L^2(0,4x)}\right)\Big|_{a=n} = e^{-x}\tau_n(x)$$

double scaling limits



- Painlevé 1895, Garnier 1912: general scaling limits for Painlevé equations
- multivariate statistics (Anderson '63, Muirhead '82, ...):

$$\text{MANOVA}_n^{a,b} \xrightarrow[b\to\infty]{} \text{Wishart}_n^a \xrightarrow[a\to\infty]{} \text{Gauss}_n$$

- Forrester–Witte 01'–04: $n \to \infty$ in UE$_n$ (bulk, soft edge, hard edge)
- Borodin–Forrester '03: hard-to-soft $a \to \infty$

operator theory

integral operators operators acting on $L^2(0,s), L^2(s,\infty), \ldots$

$$F_h(s) = \det(I - K_h), \quad K_h(x,y) = K(x,y) + \sum_{j=0}^{m} L_j(x,y)h^j + h^m \underbrace{O(e^{-x-y})}_{\text{uniformly differentiable in } s}$$

$\wr\!\downarrow$ B. '24

$$F_h(s) = F(s) + \sum_{j=1}^{m} G_j(s)h^j + h^m O(e^{-2s})$$

where, with $E_j = (I - K_0)^{-1}L_j$, $G_j = F \cdot d_j$,

$$\boxed{\begin{aligned} d_1 &= -\operatorname{tr} E_1 \\ d_2 &= \tfrac{1}{2}(\operatorname{tr} E_1)^2 - \frac{1}{2}\operatorname{tr} E_1^2 - \operatorname{tr} E_2 \\ d_3 &= -\tfrac{1}{6}(\operatorname{tr} E_1)^3 + \tfrac{1}{2}\operatorname{tr} E_1 \operatorname{tr} E_1^2 - \tfrac{1}{2}\operatorname{tr}(E_1E_2 + E_2E_1) - \tfrac{1}{3}\operatorname{tr} E_1^3 + \operatorname{tr} E_1 \operatorname{tr} E_2 - \operatorname{tr} E_3 \end{aligned}}$$

computer generated with `FynCalc`: Shtabovenko et al. '16–

$\rightsquigarrow$ directly amenable for numerical evaluation w/ B.-Nyström method

expansion of Hermite kernel $\rightsquigarrow$ B. '24, Yao–Zhang '25

$$L_j = \ \mathbb{Q}\text{-linear combination of rank-one operators of the form } A^{(\mu)} \otimes A^{(\nu)}$$

perturbation theory of $\wr$ finite dimensional determinants

$$d_j(s) = \ \mathbb{Q}\text{-linear combination of minors of } (u_{\mu\nu}(s))_{\mu,\nu=1}^{\infty}$$

$$u_{\mu\nu}(s) = \Big\langle (I - K_0)^{-1} A^{(\mu)}, A^{(\nu)} \Big\rangle_{L^2(s,\infty)}$$

symmetric in $\mu\nu$

examples

$$d_1 = -u_{01} + \tfrac{1}{5} u_{22} - \tfrac{2}{5} u_{13} + \tfrac{2}{5} u_{04}$$

$$d_2 = \tfrac{1}{25} u_{09} + [\text{11 more } 1 \times 1 \text{ minors}] + \tfrac{2}{25} \begin{vmatrix} u_{03} & u_{04} \\ u_{13} & u_{14} \end{vmatrix} + [\text{9 more } 2 \times 2 \text{ minors}]$$

integrated in $\mathbb{Q}[s]^{184\times 10}$

$$d_3 = [\text{24: } 1 \times 1 \text{ minors}] + [\text{51: } 2 \times 2 \text{ minors}] + [\text{10: } 3 \times 3 \text{ minors}]$$

Airy kernel determinant $\rightsquigarrow$

$$F'/F = \sigma = u_{00} = q'^2 - sq^2 - q^4 \in \mathbb{Q}[s][q,q'] \rightsquigarrow F^{(n)}/F \in \mathbb{Q}[s][q,q']$$

more auxiliary Tracy–Widom ODEs $q_0 = q,\ q_1 = q' + u_{00}q$

$$u'_{\mu\nu} = q_\mu q_\nu, \quad q_\nu \in \mathbb{Q}[s][q_j|_{j\leqslant\nu-2}, u_{jk}|_{j+k\leqslant\nu-1}], \quad q_0 = q, \quad q_1 \in \mathbb{Q}[s][q,q']$$

integration recursively solving linear systems over $\mathbb{Q}[s]$ — Shinault–Tracy '11, B. '24

$$u_{\mu\nu} \in \mathbb{Q}[s][q,q'] \rightsquigarrow d_j \in \mathbb{Q}[s][q,q']$$

multilinear structure solving linear systems over $\mathbb{Q}[s]$ — Shinault–Tracy '11, B. '24

checked up to $j = 10$

$$\boxed{G_j = \mathbb{Q}[s]\text{-linear combination of } F', \ldots, F^{(2n)}(s)}$$

It works, but why so? $G_j = F \cdot d_j$

$$G_1 = \frac{s^2}{5}F' - \frac{3}{10}F''$$

$$G_2 = -\left(\frac{141}{350} + \frac{8s^3}{175}\right)F' + \left(\frac{39s}{175} + \frac{s^4}{50}\right)F'' - \frac{3s^2}{50}F''' + \frac{9}{200}F^{(4)}$$

$$G_3 = \left(\frac{2216s}{7875} + \frac{148s^4}{7875}\right)F - \left(\frac{53s^2}{210} + \frac{8s^5}{875}\right)F'' + \left(\frac{10403}{31500} + \frac{51s^3}{875} + \frac{s^6}{750}\right)F''' - \left(\frac{117s}{1750} + \frac{3s^4}{500}\right)F^{(4)} + \frac{9s^2}{1000}F^{(5)} - \frac{9}{2000}F^{(6)}$$

similar structure for $\beta = 1, 4$, Laguerre ensembles, hard-to-soft edge limit (B. '24–'25)

finite-size correction term $j = 1$

(a) Forrester–Mays '15: first order perturbation analysis of σ-PVI

(b) Forrester–Shen '25:

- matching a multilinear ansatz to the small-s expansion $\rightsquigarrow$ $\boxed{G_1 = -\dfrac{s^2}{12} F''}$
- proof: check that the result satisfies (a)

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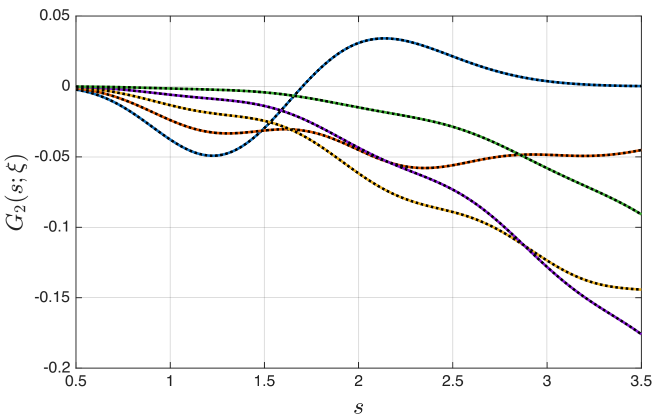
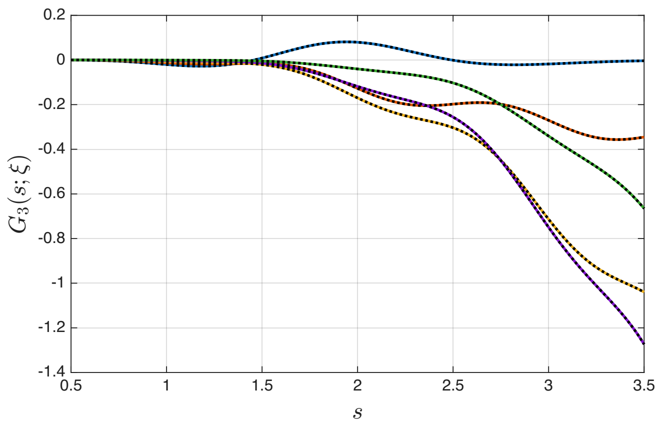
more terms
B. '25 — ~~no proof yet~~ — proof by integrating auxiliary Tracy–Widom ODEs

multilinear ansatz $\rightsquigarrow$ solving a linear system over $\mathbb{Q}[\pi]$ $\rightsquigarrow$ uniquely

$$G_2 = \frac{s^4 F^{(4)}}{288} - \frac{s^3 F'''}{360} - \left(\frac{\pi^2 s^4}{360} + \frac{s^2}{720}\right) F''$$

$$G_3 = -\frac{s^6 F^{(6)}}{10368} + \frac{s^5 F^{(5)}}{4320} + \left(\frac{\pi^2 s^6}{4320} - \frac{107 s^4}{181440}\right) F^{(4)} + \left(\frac{s^3}{11340} - \frac{\pi^2 s^5}{2835}\right) F^{(3)} + \left(-\frac{\pi^4 s^6}{5670} - \frac{\pi^2 s^4}{11340} + \frac{13 s^2}{12960}\right) F''$$





A very merry 23 674th un-birthday to you, Peter!

'They gave it me,' Humpty Dumpty continued thoughtfully, as he crossed one knee over the other and clasped his hands round it, 'they gave it me—for an un-birthday present.'

'I beg your pardon?' Alice said with a puzzled air.

'I'm not offended,' said Humpty Dumpty.

'I mean, what *is* an un-birthday present?'

'A present given when it isn't your birthday, of course.'

Lewis Carroll. *Through the Looking-Glass, and What Alice Found There* (1871)

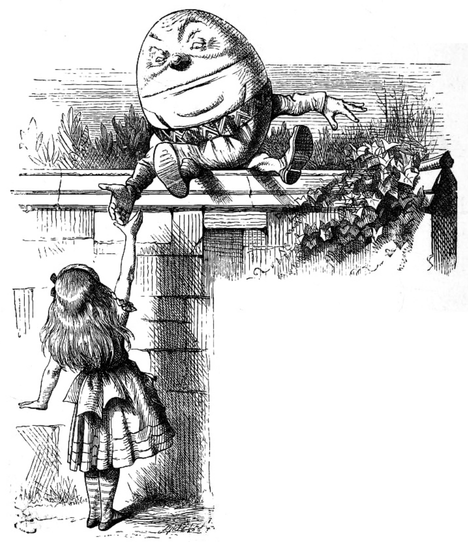


Illustration by John Teniel

aide-memoire: $e/(e^{1/\pi} + e^2 + e) \approx 0.23674\,000$