Law of fractional logarithm for random matrices

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Log-gases in Caeli Australi

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Classical laws for iid sum

Consider a sequence of iid r.v.s $X_1, X_2, \ldots$ with mean 0 and variance 1. Let

$$S_n = \sum_{i=1}^{n} X_i$$

SLLN

$$\frac{S_n}{n} \xrightarrow{\text{a.s.}} 0$$

CLT

$$\frac{S_n}{\sqrt{n}} \Longrightarrow N(0,1)$$

LIL

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n \log\log n}} = 1, \qquad \liminf_{n\to\infty} \frac{S_n}{\sqrt{2n \log\log n}} = -1, \qquad \text{a.s.}$$

Laws for extreme random matrix eigenvalue

Consider a doubly infinite array of independent (up to symmetry $x_{ij} = \bar{x}_{ji}$) real ($\beta = 1$) or complex ($\beta = 2$) r.v.s $X = (x_{ij})_{i,j\in\mathbb{N}}$: (i) mean 0, (ii) $\mathbb{E}x_{ij}^2 = 1 + \delta_{ij}$ ($\beta = 1$) and $\mathbb{E}|x_{ij}|^2 = 1, \mathbb{E}x_{ij}^2 = \delta_{ij}$ ($\beta = 2$), (iii) existence of all moments.

Wigner Minor Process Let $X^{(N)}$ be the $N \times N$ upper left corner of X. Denote

$$H^{(N)} = \frac{1}{\sqrt{N}} X^{(N)}, \qquad N = 1, 2, \ldots$$

and let $\tilde{\lambda}_i^{(N)}$ be the i-th largest eigenvalue of $H^{(N)}$.

Bai-Yin Law [Bai-Yin '88]

$$\tilde{\lambda}_1^{(N)} \xrightarrow{\text{a.s.}} 2.$$

Tracy-Widom Law [Forrester '93] [Tracy-Widom '94 '96] etc.

$$\lambda_1^{(N)} \equiv N^{2/3}(\tilde{\lambda}_1^{(N)} - 2) \Longrightarrow \mathrm{TW}_\beta.$$

Question[Kalai '13] Any logarithmic type law?

Paquette-Zeitouni's LFL for GUE

Theorem(Law of Fractional Logarithm) [Paquette-Zeitouni '17] For GUE

$$\limsup \frac{\lambda_1^{(N)}}{(\log N)^{2/3}} = \left(\frac{1}{4}\right)^{2/3}, \qquad \text{a.s.}$$

$$-c_1 \leqslant \liminf \frac{\lambda_1^{(N)}}{(\log N)^{1/3}} \leqslant -c_2, \qquad \text{a.s.}$$

Conjecture 1: $c_1 = c_2 = 4^{1/3}$

Conjecture 2: Universality (other symmetry class and entry distribution)?

Remark: The proof in [Paquette-Zeitouni '17] heavily relies on the determinantal structure of GUE minor process [Johansson-Nordenstam '06], [Forrester-Nagao '08], etc.

BBBK's lower limit for GUE

Theorem[Baslingker-Basu-Bhattacharjee-Krishnapur '24b] For GUE

$$\liminf \frac{\lambda_1^{(N)}}{(\log N)^{1/3}} = -4^{1/3}, \qquad \text{a.s.}$$

Remark Two key inputs in [BBBK '24b]

(i) A sharp lower tail estimate of $\lambda_1^{(N)}$ for all β-ensembles from [Baslingker-Basu-Bhattacharjee-Krishnapur '24a]

(ii) A decorrelation estimate via passage times in Brownian last passage percolation [Baryshnikov '01]

LFL for Wigner matrices

Theorem [B.-Cipolloni-Erdős-Henheik-Kolupaiev '25a] For Wigner minor process in either symmetric class, $\beta = 1$ for real symmetric and $\beta = 2$ for complex Hermitian, we have almost surely

$$\limsup_{N\to\infty} \frac{\lambda_1^{(N)}}{(\log N)^{2/3}} = \left(\frac{1}{2\beta}\right)^{2/3}, \qquad \liminf_{N\to\infty} \frac{\lambda_1^{(N)}}{(\log N)^{1/3}} = -\left(\frac{8}{\beta}\right)^{1/3}.$$

Remark: Our overall proof strategy is not via a comparison as there is no GOE result, although for the one point tail probability estimate we do rely on comparison with [BBBK '24a].

All Possible Limit Points

Corollary [B.-Cipolloni-Erdős-Henheik-Kolupaiev '25a] For Wigner minor process in either symmetric class, $\beta = 1$ for real symmetric and $\beta = 2$ for complex Hermitian, we have almost surely

$$\bigcap_{m=1}^{\infty} \overline{\left\{\lambda_1^{(N)}/(\log N)^{2/3} : N \geqslant m\right\}} = \left[0, \left(\frac{1}{2\beta}\right)^{2/3}\right]$$

$$\bigcap_{m=1}^{\infty} \overline{\left\{\lambda_1^{(N)}/(\log N)^{1/3} : N \geqslant m\right\}} = \left[-\left(\frac{8}{\beta}\right)^{1/3}, \infty\right]$$

where $\overline{\mathcal{E}}$ is the closure of a set $\mathcal{E} \subset \mathbb{R}$.

Key steps for classical LIL

For some sequence $a_n \to \infty$ but $a_n/\sqrt{n} \to 0$, we set the event

$$E_n := \Big\{ S_n \geqslant a_n \sqrt{n} \Big\}$$

1: One point small deviation

$$\mathbb{P}\big(E_n\big) = \exp(-a_n^2(1+o(1))/2)$$

2: Decorrelation

$$\mathbb{P}(E_n \cap E_{n+m}) = \mathbb{P}(E_n)\mathbb{P}(E_{n+m})(1+o(1)), \qquad \text{if } m \gg n$$

3: Correlation

$$E_n \approx E_{n+m}, \qquad \text{if } m \ll n$$

Small deviation of extreme eigenvalue

Proposition Consider general Wigner matrices. For any (big) $K>0$ and (small) $\varepsilon>0$, there exist $C_1 \equiv C_1(K,\varepsilon)$ and $C_2 \equiv C_2(K,\varepsilon)$, s.t.

(i) [Right tail] For any $1 \leqslant x \leqslant K(\log N)^{2/3}$,

$$C_1^{-1} \exp\Big(-\frac{2\beta}{3}(1+\varepsilon)x^{3/2}\Big) \leqslant \mathbb{P}(\lambda_1^{(N)} \geqslant x) \leqslant C_1 \exp\Big(-\frac{2\beta}{3}(1-\varepsilon)x^{3/2}\Big)$$

(ii) [Left tail] For any $1 \leqslant x \leqslant K(\log N)^{1/3}$,

$$C_2^{-1} \exp\Big(-\frac{\beta}{24}(1+\varepsilon)x^{3}\Big) \leqslant \mathbb{P}(\lambda_1^{(N)} \leqslant -x) \leqslant C_2 \exp\Big(-\frac{\beta}{24}(1-\varepsilon)x^{3}\Big)$$

Remark Part of the results are from [Erdős-Xu '23]. The rest is obtained via a comparison with [BBBK '24a], using the Green function comparison approach in [Erdős-Xu '23] for small deviation regime. Also see [Aubrun '05], [Ledoux-Rider '10], [Paquette-Zeitouni '14] etc. for earlier results.

Correlation-Decorrelation transition

Theorem[B.-Cipolloni-Erdős-Henheik-Kolupaiev '25b] Consider general real ($\beta = 1$) or complex ($\beta = 2$) Wigner minor process.

(i) For $1 \ll k \le N^{2/3-\varepsilon}$

$$\frac{N^{1/3}\big(\lambda_1^{(N)} - \lambda_1^{(N-k)}\big)}{\sqrt{k}} \sim \mathcal{N}(0, 2/\beta)\,,$$

(ii) For $N^{2/3+\varepsilon} \le k \le N^{1-\varepsilon}$, and any smooth compactly supported test-functions F, G

$$\mathbb{E}\big[F\big(\lambda_1^{(N)}\big)G\big(\lambda_1^{(N-k)}\big)\big] - \mathbb{E}\big[F\big(\lambda_1^{(N)}\big)\big]\mathbb{E}\big[G\big(\lambda_1^{(N-k)}\big)\big] \to 0.$$

Remark 1: The threshold $N^{2/3}$ is indicated by **[Forrestor-Nagao '11]**.

Remark 2: For LFL, we need above transition for small/moderate deviation regime.

Upper bound for $\limsup$: Subsequence

Aim Prove

$$\limsup_{N\to\infty} \frac{\lambda_1^{(N)}}{(\log N)^{2/3}} \leqslant c \quad a.s., \qquad \forall c > \left(\frac{1}{2\beta}\right)^{2/3}$$

$$\iff \mathbb{P}\left(\frac{\lambda_1^{(N)}}{(\log N)^{2/3}} > c \quad \text{i.o.}\right) = 0$$

Subsequence: Let $\alpha < 3$ and $N_k = [k^\alpha]$ s.t.

$$N_{k+1} - N_k \ll N_k^{2/3}.$$

Since

$$\sum_k \mathbb{P}\left[\lambda_1^{(N_k)} \geqslant c(\log N_k)^{2/3}\right] \lesssim \sum_k k^{-\alpha\frac{2\beta}{3}(1-\varepsilon)c^{3/2}} < \infty$$

By BC I

$$\limsup_{k\to\infty} \frac{\lambda_1^{(N_k)}}{(\log N_k)^{2/3}} \leqslant c.$$

Upper bound for $\limsup$: Full sequence

Lemma[Correlation estimate] For any $\delta > 0$

$$\sum_k \mathbb{P}\big(\mathcal{E}_k(\delta)\big) < \infty$$

where

$$\mathcal{E}_k(\delta) = \Big\{\exists n \in [N_{k-1}+1, N_k] : \lambda_1^{(n)} \geqslant (c+\delta)(\log n)^{2/3}\Big\} \bigcap \Big\{\lambda_1^{(N_k)} \leqslant c(\log N_k)^{2/3}\Big\}$$

Consequently

$$\limsup_{N\to\infty} \frac{\lambda_1^{(N)}}{(\log N)^{2/3}} \leqslant c + \delta.$$

Remark: Essentially, we need bound $\lambda_1^{(N_k)} - \lambda_1^{(n)}$ from below, uniformly in n.

Correlation: Key equation

Write

$$H^{(n)} = \begin{bmatrix} \sqrt{\frac{n-1}{n}} H^{(n-1)} & \mathbf{a}^{(n)} \\ (\mathbf{a}^{(n)})^* & h_{nn}^{(n)} \end{bmatrix}$$

Key Equation

$$\tilde{\lambda}_1^{(n)} = h_{nn}^{(n)} + \frac{1}{n} \sum_{\alpha=1}^{n-1} \frac{|\xi_\alpha^{(n)}|^2}{\tilde{\lambda}_1^{(n)} - \sqrt{\frac{n-1}{n}} \tilde{\lambda}_\alpha^{(n-1)}}, \qquad \xi_\alpha^{(n)} = \sqrt{n} (\mathbf{w}_\alpha^{(n-1)})^* \mathbf{a}^{(n)}$$

We rewrite it as

$$\underbrace{\tilde{\lambda}_1^{(n)}}_{\approx 2} = \underbrace{h_{nn}^{(n)}}_{\approx 0} + \frac{1}{n} \frac{|\xi_1^{(n)}|^2}{\tilde{\lambda}_1^{(n)} - \sqrt{\frac{n-1}{n}} \tilde{\lambda}_1^{(n-1)}} + \underbrace{\frac{1}{n} \sum_{\alpha=2}^{n^\varepsilon} \frac{|\xi_\alpha^{(n)}|^2}{\tilde{\lambda}_1^{(n)} - \sqrt{\frac{n-1}{n}} \tilde{\lambda}_\alpha^{(n-1)}}}_{>0} + \underbrace{\frac{1}{n} \sum_{\alpha > n^\varepsilon} \frac{|\xi_\alpha^{(n)}|^2}{\tilde{\lambda}_1^{(n)} - \sqrt{\frac{n-1}{n}} \tilde{\lambda}_\alpha^{(n-1)}}}_{\approx 1}$$

Correlation: Martingale Approach

By local law and rigidity [Erdős-Yau-Yin '12]

$$\tilde{\lambda}_1^{(n)} - \tilde{\lambda}_1^{(n-1)} \geqslant \frac{1}{n}\Big(|\xi_1^{(n)}|^2 - 1\Big) + O_{\prec}(n^{-4/3})$$

Sum up for any $n \in [N_{k-1}+1, N_k]$,

$$\begin{aligned}\tilde{\lambda}_1^{(N_k)} - \tilde{\lambda}_1^{(n)} &\geqslant \sum_{\ell=n+1}^{N_k} \frac{1}{\ell}(|\xi_1^{(\ell)}|^2 - 1) + O_{\prec}(N_k^{-2/3}) \\ &= \sum_{\ell=N_{k-1}+1}^{N_k} \frac{1}{\ell}(|\xi_1^{(\ell)}|^2 - 1) - \sum_{\ell=N_{k-1}+1}^{n} \frac{1}{\ell}(|\xi_1^{(\ell)}|^2 - 1) + O_{\prec}(N_k^{-2/3})\end{aligned}$$

Conclude the proof by maximal inequality of martingale.

Lower bound for lim sup: Decorrelation

Aim Prove

$$\limsup_{N\to\infty} \frac{\lambda_1^{(N)}}{(\log N)^{2/3}} \geqslant c \quad a.s., \qquad \forall c < \left(\frac{1}{2\beta}\right)^{2/3}$$
$$\iff \mathbb{P}\Big(\frac{\lambda_1^{(N)}}{(\log N)^{2/3}} \geqslant c \quad \text{i.o.}\Big) = 1$$

Proposition[Decorrelation estimate] For any $N_1, N_2 \in \mathbb{N}$ and any $x_1, x_2 \in [K^{-1}, K]$ with any large constant $K > 0$, we set the tail events

$$\mathcal{F}_\ell(x_\ell) := \Big\{\lambda_1^{(N_\ell)} \geqslant x_\ell (\log N_\ell)^{2/3}\Big\}, \qquad \ell = 1, 2.$$

If $N_2^{\frac{2}{3}+\varepsilon} \leqslant N_2 - N_1 \leqslant N_2^{1-\varepsilon}$ we have

$$\mathbb{P}\big(\mathcal{F}_1(x_1) \cap \mathcal{F}_2(x_2)\big) = \mathbb{P}\big(\mathcal{F}_1(x_1)\big)\mathbb{P}\big(\mathcal{F}_2(x_2)\big)(1 + O(N_2^{-\delta}))$$

Remark In the spirit of BC II, decorrelation allows one to prove the lower bound for a sufficiently independent subsequence.

Decorrelation: a dynamic approach

We rely on the dynamic approach [Cipolloni-Erdős-Schröder '23].

Dynamic of matrices

$$\mathrm{d}H_t^{(n)} = \frac{\mathrm{d}B_t^{(n)}}{\sqrt{n}}, \qquad H_0^{(n)} = H^{(n)}, \qquad n = N, N-k, \qquad k \gg N^{2/3}$$

Dynamic of minor process

$$\mathrm{d}\tilde{\lambda}_i^{(n)} = \frac{\mathrm{d}b_i^{(n)}(t)}{\sqrt{\beta n}} + \frac{1}{N}\sum_{j\neq i} \frac{1}{\tilde{\lambda}_j^{(n)}(t) - \tilde{\lambda}_i^{(n)}(t)}\mathrm{d}t, \qquad n = N, N-k$$

Here $\{b_i^{(N)}(t)\}$ and $\{b_i^{(N-k)}(t)\}$ are two collections of independent standard BM, but these two collections are dependent

$$\mathrm{d}\left[b_i^{(N)}(t), b_j^{(N-k)}(t)\right] = \left|\langle \mathbf{w}_i(H^{(N)}(t)), \mathbf{w}_j(H^{(N-k)}(t))\rangle\right|^2 \mathrm{d}t =: \Theta_{ij}^{(N,N-k)}(t)\mathrm{d}t.$$

Heuristic: $\tilde{\lambda}_1^{(N)}(t)$ and $\tilde{\lambda}_1^{(N-k)}(t)$ will be indep. after sufficiently long time ($t \gtrsim N^{-1/3+\varepsilon}$) if $\Theta_{ij}^{(N,N-k)}(t)$ is small. For the indep. of the $\tilde{\lambda}_1^{(N)}(0)$ and $\tilde{\lambda}_1^{(N-k)}(0)$, we do a Green function comparison with $\tilde{\lambda}_1^{(N)}(t)$ and $\tilde{\lambda}_1^{(N-k)}(t)$.

Eigenvector overlap

Green function

$$G^{(n)}(z) = (H^{(n)} - z)^{-1}.$$

Overlap bound For $z_a = E_a + \mathrm{i}\eta_a$, with $\eta_a \sim N^{-2/3+\varepsilon}$

$$\frac{1}{N}\mathrm{Tr}\Im G^{(N)}(z_1)\Im G^{(N-k)}(z_2) \geqslant \frac{1}{N}\frac{\eta_1}{(\tilde{\lambda}_i^{(N)} - E_1)^2 + \eta_1^2}\frac{\eta_2}{(\tilde{\lambda}_j^{(N-k)} - E_2)^2 + \eta_2^2}\Theta_{ij}^{(N,N-k)}$$

The estimate of the LHS is done by two-G local law [Cipolloni-Erdős-Schröder '22].

THANK YOU!

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HAPPY BIRTHDAY, PETER!