Correlation functions between singular values and eigenvalues

Matthias Allard

Log-gases in Caeli Australi

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arXiv:2403.19157 [Allard and Kieburg, 2024]

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Overview

Introduction

State of the art

Main results: n fixed

Numerical simulations

Main results: $n \to \infty$

Outline

Definition

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The **squared singular values** of a complex matrix $X \in \mathbb{C}^{n \times n}$ are the eigenvalues of the matrix $X^\dagger X$.

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- **Eigenvalues (EV)** of X: $\quad Z = \mathrm{diag}(z_1, \ldots, z_n) \in \mathbb{C}^n$
- **Singular values (SV)** of X: $\quad \Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_n) \in \mathbb{R}_+^n$

Relations between EV and SV

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An immediate relation between EV and SV

$$|\det(X)| = |\det(Z)| = \prod_{k=1}^{n} |z_k| = \sqrt{\det(X^\dagger X)} = \det(\Sigma) = \prod_{k=1}^{n} \sigma_k .$$

Weyl's inequalities

Ordering the EV and SV like $|z_1| \geq |z_2| \geq \ldots \geq |z_n|$ and $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n$,

From [Weyl, 1949] .

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- Upper and lower bounds

$$\sigma_1 \geq |z_1| \qquad \text{and} \qquad \sigma_n \leq |z_n|.$$

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EV & SV usually used/studied separately, but can be complementary.

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- Time Series Analysis of time-lagged matrices [Thurner and Biely, 2007, Long et al., 2023, Yao and Yuan, 2022, Loubaton and Mestre, 2021, Bhosale et al., 2018, Nowak and Tarnowski, 2017]

Outline

Question: What has already been said **probabilistically** about the relation between EV and SV when X is not Hermitian?

Bi-unitarily invariant ensembles

Random matrix ensemble$:=$ probability measure μ on a matrix space $\mathcal{M} \subset \mathbb{C}^{n \times m}$.

Definition (Bi-unitarily Invariant Ensemble)
A random matrix ensemble is said to be **bi-unitarily invariant** if for any $U_1, U_2 \in U(n)$ independent of X

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Notation/Example:
When μ_{BU} admits a density we write $d\mu_{BU}(X) = f_{BU}(X)dX$.

Example: $\exp\big(-\operatorname{Tr}\big(X^\dagger X\big)\big)dX$ (GinUE),
& all measures of the form $d\mu(X) = f(\operatorname{Tr}\big(P(X^\dagger X)\big), \det\big(Q(X^\dagger X)\big))dX$, with P, Q converging series.

Known results

Infinite matrix size $n \to \infty$:

- For bi-unitarily invariant ensembles, the complex spectrum is **isotropic**. (Follows from the definition)

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- The **Single Ring Theorem**: limiting distributions μ_{EV} and μ_{SV} exist (BU ensembles). $\mathrm{supp}(\mu_{\mathrm{EV}})$ = single annulus or disk. [Feinberg and Zee, 1997a, Feinberg and Zee, 1997b, Guionnet et al., 2009, Rudelson and Vershynin, 2014]

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- The **Haagerup-Larsen Theorem** gives the map $\mu_{\mathrm{EV}} \leftrightarrow \mu_{\mathrm{SV}}$ (BU ensembles). [Haagerup and Larsen, 2000]

Finite matrix size n:

- The **SEV transform** = bijection $f_{\mathrm{EV}} \leftrightarrow f_{\mathrm{SV}}$ (BU ensembles). [Kieburg and Kösters, 2016]

The aim

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1. Study the interaction between j EV and k SV using the SEV transform
   - Starting with 1 EV and 1 SV
2. Exploit those results to study the limit $n \to \infty$

j, k-point correlation

Definition (j, k-point correlation function)
The j, k-point correlation between j eigenvalues and k squared singular values is

$$f_{j,k}(z_1, \ldots, z_j; a_1, \ldots, a_k) := \int_{\mathbb{R}_+^{n-k}} \int_{\mathbb{C}^{n-j}} f_{n,n}(z; a) dz_{j+1} \ldots dz_n da_{k+1} \ldots da_n.$$

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$$f_{j,k}(r_1, \ldots, r_j; a_1, \ldots, a_k) := \int_{\mathbb{R}_+^{n-k}} \int_{\mathbb{R}_+^{n-j}} f_{n,n}(r; a) dr_{j+1} \ldots dr_n da_{k+1} \ldots da_n.$$

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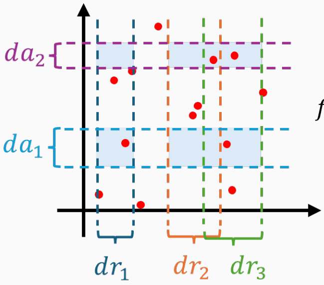

$f_{3,2}(r_1, r_2, r_3; a_1, a_2) dr_1 dr_2 dr_3 da_1 da_2$

Simple case: $n = 2$

Proposition:
For 2×2 bi-unitarily invariant matrices, the joint probability measure on the squared eigenradii and squared singular value is

$$\begin{aligned} & d\mu_{2,2}(r_1, r_2; a_1, a_2) \\ =& \Theta\left(\max\{a_1, a_2\} - \max\{r_1, r_2\}\right) \Theta\left(\min\{r_1, r_2\} - \min\{a_1, a_2\}\right) \\ & \times \frac{f_{\mathrm{SV}}(a_1, a_2)}{2|a_1 - a_2|} (r_1 + r_2)\, \delta(r_1 r_2 - a_1 a_2) dr_1 dr_2 da_1 da_2. \end{aligned}$$

with Θ the Heaviside step function, $\Theta(x)$ is 1 if $x \geq 0$, 0 otherwise.

Outline

Starting point: commutative diagram

$$\begin{tikzcd} \mathrm{L}^{1,BU}(\mathbb{C}^{n\times n}) \ni & f_{BU} \arrow[r, "\mathcal{I}_{SV}"] \arrow[d, "\mathcal{I}_{EV}"'] & f_{SV} \in \mathrm{L}^{1,SV}(\mathbb{R}_+^n) = \mathrm{L}^{1,\mathrm{Sym}}(\mathbb{R}_+^n) \arrow[dl, "\mathcal{R}"] \\ \mathrm{L}^{1,\mathrm{Sym}}(\mathbb{C}^n) \supset \mathrm{L}^{1,EV}(\mathbb{C}^n) \ni & f_{EV} & \end{tikzcd}$$

Extracted from [Kieburg and Kösters, 2016] .

Remark
$f_{BU} = \mathcal{I}_{SV}^{-1} f_{SV}$. We associate the bi-unitarily invariant ensemble with the underlying ensemble of the SV.

Starting point: SEV transform

Theorem (Map between f_{EV} and f_{SV}, [Kieburg and Kösters, 2016])
For a bi-unitarily invariant density f_{BU}, the underlying densities on the EV and squared SV are related by the bijective map $\mathcal{R}$

$$
\begin{aligned}
f_{EV}(z) &= \mathcal{R} f_{SV}(z) \\
&= \frac{\prod_{j=0}^{n-1} j!}{(n!)^2 \pi^n} |\Delta_n(z)|^2 \int_{\mathcal{C}(n)} \left[\prod_{k=1}^{n} \frac{ds_k}{2\pi i}\right] \mathrm{Perm}[|z_b|^{-2s_c}]_{b,c=1}^{n} \\
&\quad \times \int_{\mathbb{R}_+^n} \left[\prod_{j=1}^{n} \frac{da_j}{a_j}\right] f_{SV}(a) \frac{\det[{a_b}^{s_c}]_{b,c=1}^{n}}{\Delta_n(s)\Delta_n(a)},
\end{aligned}
$$

where $|z|^2 = \mathrm{diag}\left(|z_1|^2, \ldots, |z_n|^2\right)$*.* $a = \mathrm{diag}(a_1, \ldots, a_n) = \Sigma^2$*. For an n-dimensional vector x,* $\Delta_n(x) = \det\left[x_j^{k-1}\right]_{j,k=1}^{n} = \prod_{1\le j<k\le n}(x_k - x_j)$*.* $\mathcal{C}(n) = \bigtimes_{k=1}^{n}(k + i\mathbb{R})$*.*

Conditional density $\rho_{\mathrm{EV}}(.|a)$ for bi-unitarily invariant ensembles

Theorem (Conditional density $\rho_{\mathrm{EV}}(.|a)$ for general f_{BU})
Let $n > 2$, $\mathcal{C}_j = j + i\mathbb{R}$, $\tau(j) = (1, \ldots, j-1, j+1, \ldots, n)$ and $a = (a_1, \ldots, a_n) \in \mathbb{R}^n_{+*}$ pairwise distinct, the level density $\rho_{\mathrm{EV}}(.|a)$ of the squared eigenradii conditioned under fixed a is

$$\rho_{\mathrm{EV}}(r|a) = \frac{1}{n}\left(\prod_{p=0}^{n-1} p!\right)\sum_{j=1}^{n}\int_{\mathcal{C}_j}\frac{ds}{2\pi i}r^{j-1-s}\frac{\det\begin{bmatrix} a_b^{s-1} \\ a_b^{\tau_c(j)-1}\end{bmatrix}_{\substack{b=1,\ldots,n \\ c=1,\ldots,n-1}}}{\Delta_n(s,\tau(j))\Delta_n(a)}$$

$$= \frac{1}{n}\partial_r \frac{\det\left[\begin{array}{c|c} 0 & -\left(1-\frac{a_b}{r}\right)^{n-1}\Theta(r-a_b) \\ \hline 1 & \binom{n-1}{c-1}\left(-\frac{a_b}{r}\right)^{c-1}\end{array}\right]_{b,c=1,\ldots,n}}{\det\left[\binom{n-1}{c-1}\left(-\frac{a_b}{r}\right)^{c-1}\right]_{b,c=1,\ldots,n}}. \qquad (1)$$

Theorem ($1,1$-point function for general f_{BU})
For $n > 2$, $\mathcal{C}_j = j + i\mathbb{R}$ and $\tau(j) = (1, \ldots, j-1, j+1, \ldots, n)$. Let f_{SV} density on squared SV of BU ensemble f_{BU}. The $1,1$-point correlation function $f_{1,1}$ for a squared eigenradius r_1 and one squared singular value a_1 is

$$f_{1,1}(r_1; a_1) = C_n \sum_{j=1}^{n} \int_{\mathcal{C}_j} \frac{ds}{2\pi i} r_1^{j-1-s} \int_{\mathbb{R}_+^{n-1}} f_{SV}(a) \frac{\det \begin{bmatrix} a_b^{s-1} \\ a_b^{\tau_c(j)-1} \end{bmatrix}_{\substack{b=1,\ldots,n \\ c=1,\ldots,n-1}}}{\Delta_n(s, \tau(j)) \Delta_n(a)} \prod_{b=2}^{n} da_b,$$

and the 1-point function of the squared eigenradii is

$$\rho_{EV}(r_1) := f_{1,0}(r_1) = C_n \sum_{j=1}^{n} r_1^{j-1} \mathcal{M}^{-1} \left[\mathcal{S} f_{SV}(., \tau(j))\right](r_1).$$

Polynomial ensembles

Definition (Polynomial ensembles)

[Kuijlaars and Stivigny, 2014]

Ensemble on $\mathbb{R}_+^n$ with j.p.d.f.

$$f_{SV}(a) \propto \Delta_n(a) \det \left[w_{k-1}(a_j)\right]_{j,k=1}^{n} = \underbrace{\det \left[K_n(a_j, a_k)\right]_{j,k=1}}_{\text{DPP}},$$

where $w_0, \ldots, w_{n-1}$ are weight functions on $\mathbb{R}_+$ s.t. $f_{SV} \geq 0$.

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Definition (Pólya ensemble)

[Förster et al., 2020]

A polynomial ensemble such that $\exists\, w$ s.t.

$$w_k(x) = (-x\partial_x)^k w(x) \in \mathrm{L}^1(\mathbb{R}_+) \qquad k = 0, \ldots, n-1.$$

Wonderful properties of Pólya ensembles

Properties
$X, Y \in \mathbb{C}^{n\times n}$, whose SV $\hookrightarrow$ Pólya with w_X and w_Y:

1. XY has SV $\hookrightarrow$ Pólya with w_{XY}

$$w_{XY}(x) = (w_X * w_Y)(x) := \int_0^\infty w_X(y)\, w_Y\left(\frac{x}{y}\right) \frac{dy}{y}$$

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2. Y^{-1} has SV $\hookrightarrow$ Pólya with $w_{Y^{-1}}$

$$w_{Y^{-1}}(x) := \frac{1}{x^{n+1}} w_Y\left(\frac{1}{x}\right)$$

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3. DPP on SV and DPP on EV

Example: Laguerre (induced Ginibre)
Bi-unitarily invariant density on $\mathbb{C}^{n\times n}$:

$$f_{BU}(X) \propto \det\left(X^\dagger X\right)^{\alpha} \exp\left(-\operatorname{Tr}\left(X^\dagger X\right)\right),\ \alpha > -1.$$

Underlying density on the SV on $\mathbb{R}_+^n$:

$$f_{SV}(a) = \frac{1}{n!} \det\left[K_n(a_j, a_k)\right]_{j,k=1}^{n},$$

with

$$K_n(x,y) = \sum_{b=0}^{n-1} x^{\alpha} e^{-x} \frac{L_b^{(\alpha)}(x) L_b^{(\alpha)}(y)}{h_b^{(\alpha)}}$$

$L_b^{(\alpha)}$ Laguerre polynomial of degree b.

Example: Laguerre and Jacobi ensembles

Example: Jacobi (truncated unitary)
Bi-unitarily invariant density on $\mathbb{C}^{n\times n}$:

$$f_{BU}(X) \propto \det\left(X^\dagger X\right)^\alpha \det\left(\mathbb{1}_n - X^\dagger X\right)^\beta \Theta(\mathbb{1}_n - X^\dagger X),\ \alpha > -1, \beta > 0.$$

Underlying density on the SV on $[0,1]^n$:

$$f_{SV}(a) = \frac{1}{n!} \det\left[K_n(a_j, a_k)\right]_{j,k=1}^n,$$

with

$$K_n(x,y) = \sum_{b=0}^{n-1} x^\alpha (1-x)^\beta \frac{P_b^{(\alpha,\beta)}(1-2x) P_b^{(\alpha,\beta)}(1-2y)}{h_b^{(\alpha,\beta)}}$$

$P_b^{(\alpha,\beta)}$ Jacobi polynomial of degree b.

Theorem (Cross-covariance density for polynomial ensembles)
For a polynomial ensemble $f_{SV}(a) \propto \Delta_n(a) \det\left[w_{k-1}(a_j)\right]_{j,k=1}^n$, we have

$$f_{1,1}(r_1; a_1) = \int_0^\infty dt \int_0^{r_1} \frac{dv}{v} \varphi_n\left(\frac{v}{r_1}, t\right) \det\begin{pmatrix} K_n(v, -r_1 t) & K_n(v, a_1) - \delta(v - a_1) \\ K_n(a_1, -r_1 t) & K_n(a_1, a_1) \end{pmatrix}$$

with δ the Dirac delta function,

$$\varphi_n(x, t) := x(1-x)^{n-2}(1+t)^{-(n+2)}\left[(1 - \frac{x}{n})(1+t) - (1-x)(1+\frac{1}{n})\right].$$

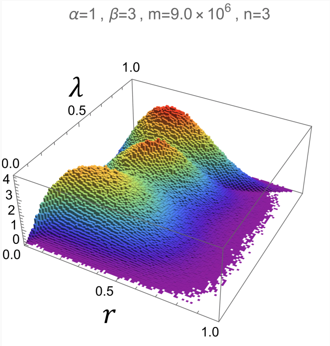
The 1-point correlation function on the squared eigenradii is

$$\rho_{EV}(r_1) := f_{1,0}(r_1) = n \int_0^\infty dt \int_0^{r_1} \varphi_n\left(\frac{v}{r_1}, t\right) K_n(v, -r_1 t) \frac{dv}{v}.$$

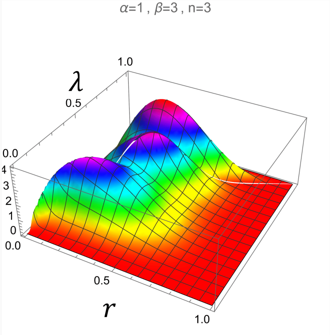
Reminder: $\rho_{SV}(r_1) := f_{0,1}(a_1) = K_n(a_1, a_1)/n$

Outline

Monte-Carlo simulations: $f_{1,1}$ Jacobi ensemble $n = 3$



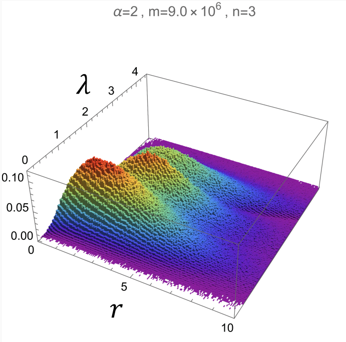
(a) Monte-Carlo simulation



(b) Numerical evaluation

Figure 1: Plots of $(r, \lambda) \mapsto 2\lambda f_{1,1}(r; \lambda^2)$ for the Jacobi ensemble $w_{Jac}(x) := x^\alpha(1-x)^{\beta+n-1}\Theta(1-x)$: $n = 3$, $(\alpha, \beta) = (1, 3)$. *m is the number of samples for the Monte-Carlo simulation.*

Monte-Carlo simulations: $f_{1,1}$ Laguerre ensemble $n = 3$



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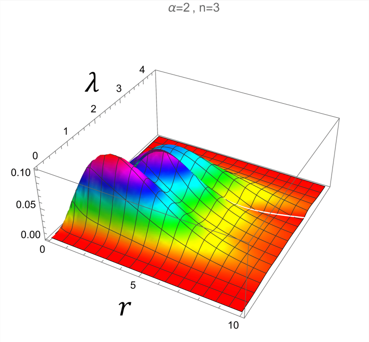
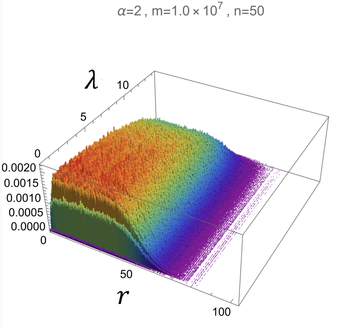


(b) Numerical evaluation

Figure 2: Plots of $(r, \lambda) \mapsto 2\lambda f_{1,1}(r; \lambda^2)$ for the Laguerre ensemble $w_{Lag}(x) = x^{\alpha} e^{-x}$: $n = 3$, $\alpha = 2$. *m is the number of samples for the Monte-Carlo simulation.*

Monte-Carlo simulations: $f_{1,1}$ Laguerre ensemble $n = 50$



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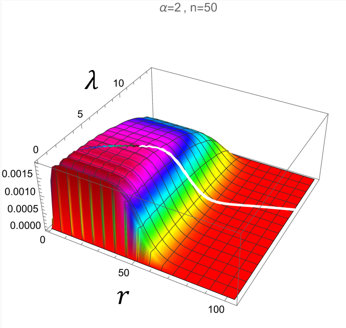


(b) Numerical evaluation

Figure 3: Plots of $(r, \lambda) \mapsto 2\lambda f_{1,1}(r; \lambda^2)$ for the Laguerre ensemble $w_{Lag}(x) = x^{\alpha} e^{-x}$: $n = 50$, $\alpha = 2$. *m is the number of samples for the Monte-Carlo simulation.*

$f_{1,1}$ Jacobi and Laguerre ensemble $n = 3$

Jacobi　　　　　　　　　　Laguerre

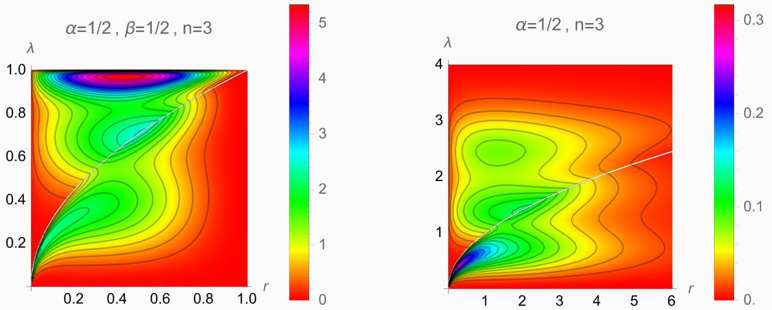


Figure 4: Plots of $(r, \lambda) \mapsto 2\lambda f_{1,1}(r; \lambda^2)$: *(Right)* Laguerre ensemble $w_{Lag}(x) = x^{\alpha} e^{-x}$: $n = 3$, $\alpha = 1/2$. *(Left)* Jacobi ensemble $w_{Jac}(x) = x^{\alpha}(1-x)^{\beta+n-1}\Theta(1-x)$: $n = 3$, $\alpha = 1/2$, $\beta = 1/2$.

Cross-covariance density

Definition
Cross-covariance density function between one squared eigenradius and one squared singular value: $cov : \mathbb{R}_+^2 \to \mathbb{R}$

$$\mathrm{cov}(x; y) := f_{1,1}(x; y) - f_{1,0}(x) f_{0,1}(y).$$

Limiting cross-covariance density around $(0,0)$: *cov*

Jacobi | Laguerre | Limit

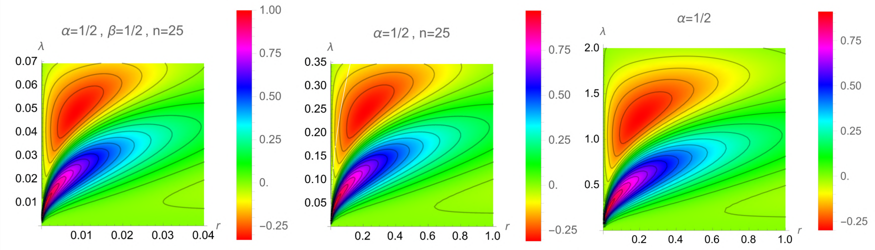


Figure 5: Plots of $(r,\lambda) \mapsto 2\lambda\mathrm{cov}(r;\lambda^2)$: $n = 25$ *(Left)* Jacobi ensemble: $\alpha = 1/2$, $\beta = 1/2$. *(Middle)* Laguerre ensemble[1]: $\alpha = 1/2$. *(Right)* Plots of $(r,\lambda) \mapsto 2\lambda\mathrm{cov}^\infty(r;\lambda^2)$: for $w^\infty(x) = x^\alpha e^{-x}$, $\alpha = 1/2$.

[1] Scaled by $n^{3/2}$

Outline

Hard edge limit around $(0,0)$: Cross-covariance for polynomial ensembles

Theorem (Cross-covariance density for polynomial ensembles)
For a polynomial ensemble, under mild assumptions, the double scaling limit $\lim_{n\to\infty} \frac{n}{\nu_n^2} \mathrm{cov}\left(\frac{r_1}{\nu_n}; \frac{a_1}{n\nu_n}\right)$ *is*

$$\mathrm{cov}_\infty(r_1; a_1) = \int_0^\infty dt \int_0^\infty \frac{dv}{v} \varphi_\infty\left(\frac{v}{r_1}, t\right) K_\infty(a_1, -r_1 t)\left[\delta(v-a) - K_\infty(v, a_1)\right]$$

with δ *the Dirac delta function,*

$$\varphi_\infty(x,t) := x(t+x-1)e^{-x}e^{-t}.$$

The 1-point correlation function on the squared eigenradii is

$$\rho_{EV,\infty}(r_1) = \int_0^\infty dt \int_0^\infty \varphi_\infty\left(\frac{v}{r_1}, t\right) K_\infty(v, -r_1 t) \frac{dv}{v}.$$

Reminder: $\rho_{SV,\infty}(a_1) = K_\infty(a_1, a_1)$,
$\mathrm{cov}_\infty(x; y) = f_{1,1;\infty}(x; y) - \rho_{EV,\infty}(x)\rho_{SV,\infty}(y)$

Hard-Soft edge around $(1/2, 1)$: Cross-covariance for Jacobi ensembles

Theorem (Cross-covariance density for polynomial ensembles)
For the Jacobi ensemble with Pólya weight $w_{Jac}(x) = x^{\alpha}(1-x)^{\beta+n-1}\Theta(1-x)$, $\alpha > -1$, $\beta > 0$, we have

$$\lim_{n\to\infty} \sqrt{n}\,\mathrm{cov}\left(\frac{1}{2} - \frac{r_1}{\sqrt{n}}; 1 - \frac{a_1}{n^2}\right) = \frac{4}{\sqrt{\pi}} e^{-4r_1^2} J_\beta(2\sqrt{a_1})^2.$$

with J_β the Bessel function of the first kind of order β.

Reminder:

$$\lim_{n\to\infty} \rho_{\mathrm{SV}}\left(1 - \frac{a_1}{n^2}\right) = J_\beta\left(2\sqrt{a_1}\right)^2 - J_{\beta-1}\left(2\sqrt{a_1}\right) J_{\beta+1}\left(2\sqrt{a_1}\right).$$

$$\lim_{n\to\infty} \rho_{\mathrm{EV}}\left(\frac{1}{2} - \frac{r_1}{\sqrt{n}}\right) = 2\,\mathrm{erfc}\left(2r_1\right),$$ [Akemann et al., 2014]

Thank you!
Happy Birthday Peter!

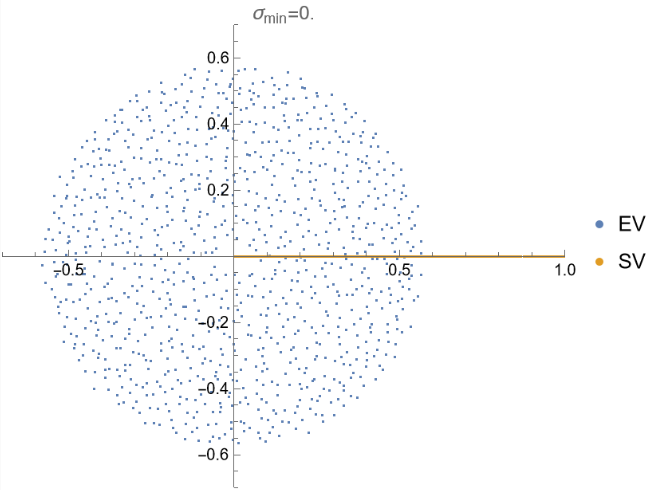
Conjecture

The conditional joint probability density of the eigenvalues is given in a distributional way by

$$d\mu_{n,n}(z|a) = \frac{\prod_{j=0}^{n-1} j!}{(n!)^2\pi^n} |\Delta_n(z)|^2 dz \lim_{\varepsilon\to 0} \int_{\mathcal{C}(n)} \left[\prod_{k=1}^{n} \frac{ds_k}{2\pi i} \zeta(\varepsilon \operatorname{Im}\{s_k\})\right]$$
$$\times \operatorname{Perm}[|z_b|^{-2s_c}]_{b,c=1}^{n} \frac{\det\left[a_b^{s_c-1}\right]_{b,c=1}^{n}}{\Delta_n(s)\Delta_n(a)}.$$

Remaining difficulty: $\mathcal{C}(n) = \bigtimes_{k=1}^{n}(k + i\mathbb{R})$

$$\lim_{\varepsilon\to 0} \int_{\mathcal{C}(n)} \left[\prod_{k=1}^{n} \frac{ds_k}{2\pi i} \zeta(\varepsilon \operatorname{Im}\{s_k\})\right] \operatorname{Perm}[r_b^{-s_c}]_{b,c=1}^{n} \frac{\det\left[a_b^{s_c-1}\right]_{b,c=1}^{n}}{\Delta_n(s)\Delta_n(a)}$$
$$= \mathcal{M}_S^{-1}\left[\frac{\det\left[a_b^{s_c-1}\right]_{b,c=1}^{n}}{\Delta_n(s)\Delta_n(a)}\right](r) = ?$$



σ_{min}=0.
0.6
0.4
0.2
−0.2
−0.4
−0.6
−0.5
0.5
1.0
EV
SV

cov Jacobi and Laguerre ensemble $n = 3$

Jacobi　　　　Laguerre

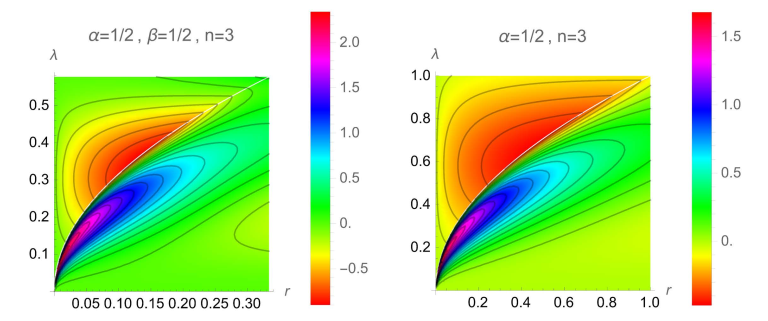


Figure 6: Plots of $(r, \lambda) \mapsto 2\lambda\text{cov}(r; \lambda^2)$: *(Right)* Laguerre ensemble[2]: $n = 3$, $\alpha = 1/2$. *(Left)* Jacobi ensemble: $n = 3$, $\alpha = 1/2$, $\beta = 1/2$.

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[2]Scaled by $n^{3/2}$

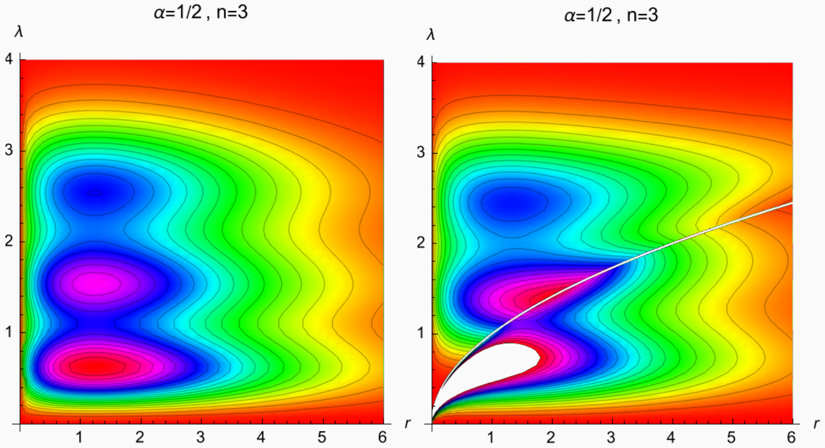
j, k-point correlation measure

Definition (j, k-point correlation measure function)
Let $n \in \mathbb{N}$, $j, k \in [\![0, n]\!]$. Let f_{BU} be the probability density function of the random matrix $X \in G = \mathrm{GL}(n, \mathbb{C})$. Denoting the squared eigenradii of X by $\{r_l(X)\}_{l=1}^n$ and its squared singular values by $\{a_l(X)\}_{l=1}^n$, then the j, k-point correlation measure $\mu_{j,k}$ is defined weakly by the relation

$$\mathbb{E}\left[\frac{(n-j)!(n-k)!}{(n!)^2} \sum_{\substack{1 \leq l_1, \ldots, l_j, p_1, \ldots, p_k \leq n \\ l_\alpha \overset{\alpha \neq \beta}{\neq} l_\beta,\ p_\alpha \overset{\alpha \neq \beta}{\neq} p_\beta}} \phi(r_{l_1}(X), \ldots, r_{l_j}(X); a_{p_1}(X), \ldots, a_{p_k}(X))\right]$$

$$= \int_{\mathbb{R}_+^{j+k}} d\mu_{j,k}(r_1, \ldots, r_j; a_1, \ldots, a_k)\phi(r_1, \ldots, r_j; a_1, \ldots, a_k)$$

for any continuous bounded function $\phi \in C_b\left(\mathbb{R}_+^{j+k}\right)$.



α=1/2 , n=3
λ
r
1
2
3
4
5
6
α=1/2 , n=3
λ
r

Definition (Singular Values)

$$\forall X \in \mathbb{C}^{n\times n},\ \exists\, \Sigma \in \mathbb{R}_+^n,\ U, V \in \mathrm{U}(n), \quad s.t. \quad X = U\Sigma V^\dagger$$

The entries of $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ are the **singular values (SV)** of the matrix X.

Definition

Definition (Singular Values)

$$\forall X \in \mathbb{C}^{n\times n},\ \exists\,\Sigma \in \mathbb{R}^n_+,\ U, V \in \mathrm{U}(n), \quad s.t. \quad X = U\Sigma V^\dagger$$

The entries of $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ are the **singular values (SV)** of the matrix X.

Definition (Schur Decomposition)

$$\forall X \in \mathbb{C}^{n\times n},\ \exists\, Z \in \mathbb{C}^n,\ T \in \mathrm{T}(n),\ U \in \mathrm{U}(n), \quad s.t. \quad X = UZTU^\dagger,$$

where $\mathrm{T}(n)$ the group of upper unitriangular matrices. The entries of $Z = \mathrm{diag}(z_1, \ldots, z_n)$ are the **eigenvalues (EV)** of the matrix X.

Definition (Mellin transform)

$$\mathcal{M}f(s) = \int_0^\infty dx\; x^{s-1} f(x)$$

Property

$$\mathcal{M}[(-x\partial_x)^k w(x)](s) = s^k \mathcal{M}w(s)$$

Outline

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